Quantum lightcone fluctuations in theories with extra dimensions

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Abstract

The effects of small extra dimensions upon quantum fluctuations of the lightcone are examined. We argue that compactified extra dimensions modify the quantum fluctuations of gravitational field so as to induce lightcone fluctuations. This phenomenon can be viewed as being related to the Casimir effect. The observable manifestation of the lightcone fluctuations is broadening of spectral lines from distant sources. In this paper, we further develop the formalism used to describe the lightcone fluctuations, and then perform explicit calculations for several models with flat extra dimensions. In the case of one extra compactified dimension, we find a large effect which places severe constraints on such models. When there is more than one compactified dimension, the effect is much weaker and does not place a meaningful constraint. We also discuss some brane worlds scenarios, in which gravitons satisfy Dirichlet or Neumann boundary conditions on parallel four-dimensional branes, separated by one or more flat extra dimensions.

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I. INTRODUCTION

One of the most challenging problems in modern physics is the unification of the gravitational interaction with other known interactions in nature. Many attempts, from early Kaluza-Klein theory [1] to present supergravity and superstring theories, all involve going to higher dimensions and postulating the existence of extra spatial dimensions. The modern view regarding these postulated extra dimensions is that they are a physical reality, rather than merely a technical intermediate device for obtaining rather complicated four-dimensional theories from simpler Lagrangians in higher dimensions. If these extra dimensions really exist, one must explain why they are not seen. The usual answer is that they curl into an extremely small compactified manifold, possibly as small as the Planck length scale, $l_{pl} = 1.6 \times 10^{-33}$ cm. Therefore low-energy physics should be insensitive to them until distances of the compactification scale are being probed. In general, one has the possibility of observing the presence of the extra dimensions in a scattering experiment in which energies greater than that associated with the compactification scale are achieved. Various upper bounds have been put on the size of possible extra dimensions [2–4]. For example, an upper bound of ~ 1 Tev was given in orbifold compactifications of superstrings [4]. However, if only gravity propagates in the extra dimensions, the upper bound can be much larger. A recent proposal is that the fundamental scale of quantum gravity can be as low as few Tev and the observed weakness of gravity is the result of large extra dimensions in which only gravity can propagate [5]. This scenario could be realized in the context of several string models [6] in which one has a set of three-branes (3+1 dimensional spacetime) in the entire spacetime with extra dimensions. The Standard Model particles are confined to one of the branes, while gravitons propagate freely in the entire bulk. The size of extra dimensions could then be as large as 1 mm in this type of model. Extra dimensions of sufficiently large size may manifest themselves in particle colliders [7] and in the possible deviation from Newton's law at short distances [8], and they may also have implications in gauge unification [9] and cosmology [10].

However, a question arises naturally as to whether there are any lower bounds on the sizes of extra dimensions. It is the common belief that the existence of extra dimensions has no effect on low-energy physics as long as they are extremely small. Recently, we have argued that this is not the case, because of lightcone fluctuations arising from the quantum gravitational vacuum fluctuations due to compactification of spatial dimensions [11,12]. An explicit calculation was carried out in the five-dimensional prototypical Kaluza-Klein model which showed that the periodic compactification of the extra spatial dimensions gives rise to stochastic fluctuations in the apparent speed of light which grow as the compactification scale decreases and are in principle observable. Basically, the smaller the size of the compactified dimensions , the larger are the fluctuations that result. This is closely related to the Casimir effect, the vacuum energy occurring whenever boundary conditions are imposed on a quantum field. The gravitational Casimir energy in the five-dimensional case with one compactified spatial dimension was studied in [13], where a nonzero energy density was found, which tends to make the extra dimension contract. This raises the question of stability of the extra dimensions. It is possible, however, that the Casimir energy arising from the quantum gravitational field and other matter fields may be made to cancel each other [14], thus stabilizing the extra dimensions. Quantum lightcone fluctuations due to the compactification of spatial dimensions [11,12,15], although similar in nature to the Casimir effect, come solely from gravitons. Hence, no similar cancellation is to be expected.

In an earlier work [12], we examined lightcone fluctuation in a five-dimensional model with periodic compactification. We found that there seem to be observable effects which essentially rule out this model. That is, we derived a lower bound on the compactification scale which is larger than upper bounds derived from other considerations. The purpose of the present paper is threefold: to further develop the basic formalism, to provide more details of the five-dimensional calculation, and to extend the analysis to some higher dimensional models.

In Section II, we give a brief review of the formalism, discuss the observability of lightcone fluctuations, and derive the graviton two-point functions in the transverse tracefree gauge for spacetimes with an arbitrary number of dimensions (detailed calculations will given in the Appendix). In Section III, we examine light cone fluctuations in spacetimes with extra dimensions periodically compactified into a torus. The five-dimensional prototypical Kaluza-Klein model, of which some results have already been reported, will be studied in great detail. Higher dimensions up through 11 will also be discussed. Section IV deals with light cone fluctuations in the brane-world scenario, as motivated by a recent proposal of extra dimensions of macroscopic size. We will summarize and conclude in Section V.

II. OBSERVABILITY OF LIGHT CONE FLUCTUATIONS AND THE GRAVITON TWO-POINT FUNCTION IN THE TT GAUGE

To begin, let us examine a $d = 4 + n$ dimensional spacetime with n extra dimensions. Consider a flat background spacetime with a linearized perturbation $h_{\mu\nu}$ propagating upon it , so the spacetime metric may be written as $ds^2 = (\eta_{\mu\nu} + h_{\mu\nu})dx^\mu dx^\nu = dt^2 - d\mathbf{x}^2 + h_{\mu\nu}dx^\mu dx^\nu$, where the indices μ, ν run through $0, 1, 2, 3, ..., 3+n$. Let $\sigma(x, x')$ be one half of the squared geodesic distance between a pair of spacetime points x and x', and $\sigma_0(x, x')$ be the corresponding quantity in the flat background . In the presence of a linearized metric perturbation, $h_{\mu\nu}$, we may expand $\sigma = \sigma_0 + \sigma_1 + O(h^2_{\mu\nu})$. Here σ_1 is first order in $h_{\mu\nu}$. If we quantize $h_{\mu\nu}$, then quantum gravitational vacuum fluctuations will lead to fluctuations in the geodesic separation, and therefore induce lightcone fluctuations. In particular, we have $\langle\sigma_1^2\rangle \neq 0$, since σ_1 becomes a quantum operator when the metric perturbations are quantized. The quantum lightcone fluctuations give rise to stochastic fluctuations in the speed of light, which may produce an observable time delay or advance Δt in the arrival times of pulses.

A. Observability of light cone fluctuation

Here, we shall discuss how light cone fluctuations characterized by $\langle\sigma_1^2\rangle$ are related to physical observable quantities. For this purpose, let us consider the propagation of light pulses between a source and a detector separated by a distance r on a flat background with quantized linear perturbations. For a pulse which is delayed or advanced by time Δt, which is much less than r, one finds

$$\sigma = \sigma_0 + \sigma_1 + ... = \frac{1}{2}[(r + \Delta t)^2 - r^2] \approx r\Delta t\,. \tag{1}$$

Square the above equation and take the average over a given quantum state of gravitons $|\phi\rangle$ (e.g. the vacuum states associated with compactification of spatial dimensions),

$$\Delta t_\phi^2 = \frac{\langle\phi|\sigma_1^2|\phi\rangle}{r^2}\,. \tag{2}$$

This result is, however, divergent due to the formal divergence of $\langle\phi|\sigma_1^2|\phi\rangle$. One can define an observable Δt_{obs} by subtracting from Eq. (2) the corresponding quantity, Δt_0^2, for the vacuum state as follows

$$\Delta t_{obs}^2 = |\Delta t_\phi^2 - \Delta t_0^2| = \frac{|\langle\phi|\sigma_1^2|\phi\rangle - \langle 0|\sigma_1^2|0\rangle|}{r^2} \equiv \frac{|\langle\sigma_1^2\rangle_R|}{r}\,. \tag{3}$$

Here we take the absolute value of the difference between Δt_ϕ^2 and Δt_0^2, because the observable quantity Δt_{obs}^2 has to be a positive real number. Note that we can also get this result from the gravitational quantum average of the retarded Green's function $\langle G_{ret}(x, x')\rangle$ when $\langle\sigma_1^2\rangle_R > 0$ [15]. Therefore, the root-mean-squared deviation from the classical propagation time is given by

$$\Delta t_{obs} = \frac{\sqrt{|\langle\sigma_1^2\rangle_R|}}{r}\,. \tag{4}$$

At this point, a question may arise as to whether the formal procedure of taking the absolute value in deriving the relation between Δt and $\langle\sigma_1^2\rangle_R$, Eq. (4), is a reasonable one, or whether a meaningful relation between Δt and $\langle\sigma_1^2\rangle_R$ can be established only when $\langle\sigma_1^2\rangle_R > 0$. We shall argue that a result essentially the same as Eq. (4) can be obtained by the following analysis which avoids the sign problem. Instead of squaring the Eq. (1), we take the fourth power of both sides and average over a quantum vacuum state to yield

$$\Delta t_\phi^4 = \frac{\langle\phi|\sigma_1^4|\phi\rangle}{r^4}\,. \tag{5}$$

We can regularize $\langle\phi|\sigma_1^4|\phi\rangle$ by normal ordering and define

$$\langle\phi|\sigma_1^4|\phi\rangle_R = \langle\phi| : \sigma_1^4 : |\phi\rangle\,. \tag{6}$$

For a free field ψ and a quantum vacuum state one can show by use of Wick's theorem that

$$\langle : \psi^4 :\rangle = 3\langle : \psi^2 :\rangle^2 = 3\langle\psi^2\rangle_R^2\,. \tag{7}$$

Therefore one finds that

$$\Delta t_{obs} = \frac{3^{1/4}\sqrt{|\langle\sigma_1^2\rangle_R|}}{r} \approx \frac{\sqrt{|\langle\sigma_1^2\rangle_R|}}{r}\,. \tag{8}$$

This is essentially the same as Eq. (4), apart from a dimensionless factor of order unity. We should note that there may be other ways to define the quartic operator, σ_1^4. One possibility is to let $\sigma_1^4 = (: \sigma_1^2 :)^2$, provided that the integrals involved can be defined. Both of these definitions were discussed in Ref. [16]. There it was found in some model cases that the two definitions yield the same result, apart from numerical factors of order unity, which are not important for the present purposes.

Note that Δt is the ensemble averaged deviation, not necessarily the expected variation in flight time, δt, of two pulses emitted close together in time. The latter is given by Δt only when the correlation time between successive pulses is less than the time separation of the pulses. This can be understood physically as due to the fact that the gravitational field may not fluctuate significantly in the interval between the two pulses. This point is discussed in detail in Ref. [17]. These stochastic fluctuations in the apparent velocity of light arising from quantum gravitational fluctuations are in principle observable, since they may lead to a spread in the arrival times of pulses from distant astrophysical sources, or the broadening of the spectral lines. Lightcone fluctuations and their possible astrophysical observability have been recently discussed in a somewhat different framework in Refs. [18,19].

B. An Alternative Derivation of Δt

In order to find Δt in a particular situation, we need to calculate the quantum expectation value $\langle \sigma_1^2 \rangle_R$ in any chosen quantum state $|\psi\rangle$, which can be shown to be given by [1] [15,11]

$$\langle \sigma_1^2 \rangle_R = \frac{1}{8} (\Delta r)^2 \int_{r_0}^{r_1} dr \int_{r_0}^{r_1} dr' \; n^\mu n^\nu n^\rho n^\sigma \; G^R_{\mu\nu\rho\sigma}(x, x') \,. \tag{9}$$

Here $dr = |d\mathbf{x}|$, $\Delta r = r_1 - r_0$ and $n^\mu = dx^\mu / dr$. The integration is taken along the null geodesic connecting two points x and x', and

$$G^R_{\mu\nu\rho\sigma}(x, x') = \langle \psi | h_{\mu\nu}(x) h_{\rho\sigma}(x') + h_{\mu\nu}(x') h_{\rho\sigma}(x) | \psi \rangle \tag{10}$$

is the graviton Hadamard function, understood to be suitably renormalized. The gauge invariance of Δt, as given by Eq. (4), has been analyzed recently [11].

In this subsection, we wish to rederive Δt using the geodesic deviation equation. This derivation allows us to see the gauge invariance more clearly, and to discuss the issue of Lorentz invariance of lightcone fluctuations. Let us consider a pair of timelike geodesics with tangent vector u^μ, and n^μ as a unit spacelike vector pointing from one geodesic to the other (See Fig. 1).

[1] Although the derivations there were given in 3+1 dimensions, the generalization to arbitrary dimensions is straightforward.

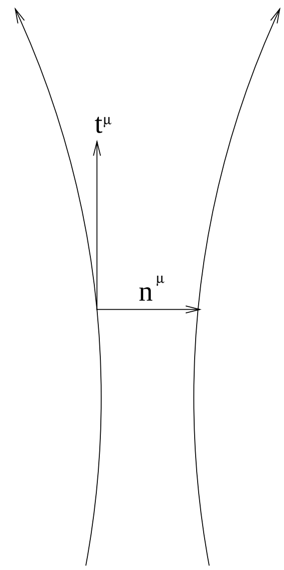

FIG. 1. A pair of nearby timelike geodesics. Here u^μ is a tangent vector along the geodesic, while n^μ is a unit spacelike vector pointing from one geodesic to the other.

The geodesic deviation equation is given by

$$\frac{D^2 n^\mu}{d\tau^2} = -R^\mu_{\alpha\nu\beta} u^\alpha n^\nu u^\beta \,, \tag{11}$$

where $R^\mu_{\alpha\nu\beta}$ is the Riemann tensor. The relative acceleration per unit proper length of particles on the neighboring geodesics is

$$\alpha \equiv n_\mu \frac{D^2 n^\mu}{d\tau^2} = -R_{\mu\alpha\nu\beta} n^\mu u^\alpha n^\nu u^\beta \,. \tag{12}$$

Thus if ds is the spatial distance between the two particles, then $\alpha\, ds$ is their relative acceleration. It follows that the relative change in displacement of the two particles after a proper time T is

$$ds \int_0^T d\tau \int_0^\tau d\tau' \alpha(\tau', 0) \,, \tag{13}$$

Now consider the case of two observers (particles) separated by a finite initial distance s_0 as illustrated in Fig. (2).

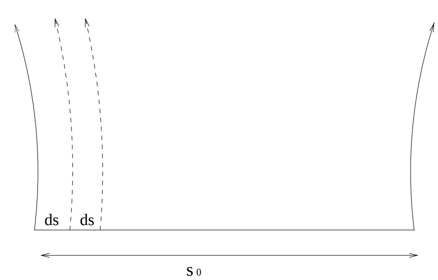

FIG. 2. Two timelike geodesics separated by a finite interval containing an infinite number of nearby geodesics .

We can find the relative change in displacement of these two observers by integrating on s:

$$\Delta s = \int_0^{s_0} ds \int_0^T d\tau \int_0^\tau d\tau' \alpha(\tau', 0)\,. \tag{14}$$

This is the relative displacement measured at the same moment of proper time for both observers.

Let us now consider a light signal sent from one observer to the other. If $\alpha = 0$, the distance traveled by the light ray is s_0. When $\alpha \neq 0$, this distance becomes $s_0 + \Delta s$, where now

$$\Delta s = \int_0^{s_0} ds \underbrace{\int_0^s d\tau \int_0^\tau d\tau' \alpha(\tau', s)}\,. \tag{15}$$

Here the under-braced integral is the displacement per unit s of a pair of observers at a distance s from the source. The domain of the final two integrations is illustrated in Fig. (3).

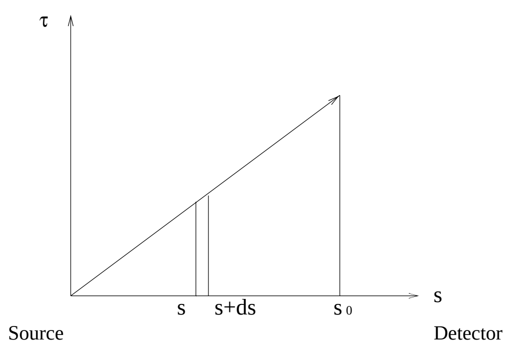

FIG. 3. The displacement, Δs, between a source and a detector is given by an integration within the triangular region.

If gravity is quantized, the Riemann tensor will fluctuate around an average value of zero due to quantum gravitational vacuum fluctuations. This leads to $\langle\alpha\rangle = 0$, and hence $\langle\Delta s\rangle = 0$. Notice here that α becomes a quantum operator when metric perturbations are quantized. However, in general, $\langle(\Delta s)^2\rangle \neq 0$, and we have

$$\langle(\Delta s)^2\rangle = \int_0^{s_0} ds_1 \int_0^{s_0} ds_2 \int_0^{s_1} d\tau_1 \int_0^{\tau_1} d\tau_1' \int_0^{s_2} d\tau_2 \int_0^{\tau_2} d\tau_2' \, \langle\alpha(\tau_1', s_1)\alpha(\tau_2', s_2)\rangle\,. \tag{16}$$

Thus the root-mean-squared fluctuation in the flight path is $\sqrt{\langle(\Delta s)^2\rangle}$, which can also be understood as a fluctuation in the speed of light. It entails an intrinsic quantum uncertainty in the measurement of distance. Therefore, spacetime becomes fuzzy at a scale characterized by $\sqrt{\langle(\Delta s)^2\rangle}$. The integrand in Eq. (16) is obviously invariant under any coordinate transformation while the integral is gauge invariant within the linear approximation.

We now wish to show that this gauge-invariant quantity is the same as Eq. (4) when calculated in the transverse-tracefree (TT) gauge. Choose a coordinate system where the source and the detector are both at rest, and suppose that the light ray propagates in the x-direction, then we have

$$u^\mu = (1, 0, 0, 0)\,, \tag{17}$$

$$n^\mu = (0, 1, 0, 0)\,, \tag{18}$$

and

$$\alpha = R_{xtxt} = -\frac{1}{2} h_{xx,tt}\,. \tag{19}$$

Substitution of the above results into Eq. (16) leads to

$$\begin{aligned}\langle(\Delta s)^2\rangle &= \int^r dx_1 \int_0^r dx_2 \int_0^{x_1} dt_1 \int_0^{t_1} dt_1' \int_0^{x_2} dt_2 \int_0^{t_2} dt_2' \,\langle\alpha(t_1', s_1)\alpha(t_2', s_2)\rangle \\ &= \frac{1}{4}\int_0^r dx_1 \int_0^r dx_2 \,\langle h_{xx}(x_1, x_1) h_{xx}(x_2, x_2)\rangle = \frac{1}{r}\langle\sigma_1^2\rangle\,,\end{aligned} \tag{20}$$

where we have set $s_0 = r$ and used the fact that along the light ray $x = t$. Thus, one has

$$\Delta t = \frac{\sqrt{\langle\sigma_1^2\rangle}}{r} = \sqrt{\langle(\Delta s)^2\rangle} \tag{21}$$

which also demonstrates the gauge-invariance of Δt.

Now we wish to discuss the rather subtle issue of the relation of lightcone fluctuations to Lorentz symmetry. It is sometimes argued that lightcone fluctuations are incompatible with Lorentz invariance. The most dramatic illustration of this arises when a time advance occurs, that is, when a pulse propagates outside of the classical lightcone. In a Lorentz invariant theory, there will exist a frame of reference in which the causal order of emission and detection is inverted, so the pulse is seen to be detected before it was emitted. Thus the lightcone fluctuation phenomenon, if it is to exist at all, seems to be incompatible with strict Lorentz invariance.

Our view of the situation is the following: lightcone fluctuations respect Lorentz symmetry on the average, but not in individual measurements. The symmetry on the average insures that the mean lightcone be that of classical Minkowski spacetime. The average metric is that of Minkowski spacetime provided that $\langle h_{\mu\nu}\rangle = 0$. However, a particular pulse effectively measures a spacetime geometry which is not Minkowskian and not Lorentz invariant. A simple model may help to illustrate this point. Consider a quantum geometry consisting of an ensemble of classical Schwarzschild spacetimes, but with both positive and negative values for the mass parameter M. (The fact that the $M < 0$ Schwarzschild spacetime has a naked singularity at $r = 0$ need not concern us. For the purpose of this model, we can confine our discussion to a region where $r \gg |M|$.) Suppose that this ensemble has

$\langle M \rangle = 0$, but $\langle M^2 \rangle \neq 0$. It is well known that light propagation in a $M > 0$ Schwarzschild spacetime can exhibit a time delay relative to what would be expected in flat spacetime. This is the basis for the time delay tests of general relativity using radar signals sent near the limb of the sun. In the present model, however, the time difference is equally likely to be a time advance rather than a time delay. A measurement of the time difference amounts to a measurement of M. This model is Lorentz invariant on the average because $\langle M \rangle = 0$ and the average spacetime is Minkowskian. However, a specific measurement selects a particular member of the ensemble, which is generally not Lorentz invariant.

In addition to the fact that the mean metric is Minkowskian, there is another sense in which lightcone fluctuations due to compactification exhibit average Lorentz invariance. Note that Δs, and hence Δt, depends on the Riemann tensor correlation function $\langle R_{xtxt}(x_1) R_{xtxt}(x_2) \rangle$, which is invariant under Lorentz boosts along the x-axis. Thus if we were to repeat the above calculations of Δs in a second frame moving with respect to the first, the result will be the same. In both cases one is assuming that the detector is at rest relative to the source. This is a reflection of the Lorentz invariance of the spectrum of fluctuations, which is exhibited by the compactified flat spacetimes studied in this paper, but not by the Schwarzschild spacetime with a fluctuating mass.

C. Graviton two-point functions

We shall use a quantization of the linearized gravitational perturbations $h_{\mu\nu}$ which retains only physical degrees of freedom. That is, we are going to work in the TT gauge in which the gravitational perturbations have only spatial components h_{ij}, satisfying the transverse, $\partial^i h_{ij} = 0$, and tracefree, $h^i_i = 0$ conditions. Here i, j run from 1 to $3 + n = d - 1$. These $2d$ conditions remove all of the gauge degrees of freedom and leave $\frac{1}{2}(d^2 - 3d)$ physical degrees of freedom. We have

$$h_{ij} = \sum_{\mathbf{k},\lambda} [a_{\mathbf{k},\lambda} e_{ij}(\mathbf{k}, \lambda) f_{\mathbf{k}} + H.c.]. \tag{22}$$

Here H.c. denotes the Hermitian conjugate, λ labels the $\frac{1}{2}(d^2 - 3d)$ independent polarization states, $f_{\mathbf{k}}$ is the mode function, and the $e_{\mu\nu}(\mathbf{k}, \lambda)$ are polarization tensors. (Units in which $32\pi G_d = 1$, where G_d is Newton's constant in d dimensions and in which $\hbar = c = 1$ will be used in this paper.)

Let us now calculate the Hadamard function, $G_{\mu\nu\rho\sigma}(x, x')$, for gravitons in the Minkowski vacuum state in the transverse tracefree gauge. It follows that

$$G_{ijkl}(x, x') = \frac{2Re}{(2\pi)^{d-1}} \int \frac{d^{d-1}\mathbf{k}}{2\omega} \sum_{\lambda} e_{ij}(\mathbf{k}, \lambda) e_{kl}(\mathbf{k}, \lambda) e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} e^{-i\omega(t-t')} \,. \tag{23}$$

The summation of polarization tensors in the transverse tracefree gauge is (See the Appendix in Ref. [11].[2])

[2]The tensorial argument given there applies in any number of dimensions

$$\sum_{\lambda} e_{ij}(\mathbf{k},\lambda)\, e_{kl}(\mathbf{k},\lambda) = \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \delta_{ij}\delta_{kl} + \hat{k}_i\hat{k}_j\hat{k}_k\hat{k}_l$$
$$+\hat{k}_i\hat{k}_j\delta_{kl} + \hat{k}_k\hat{k}_l\delta_{ij} - \hat{k}_i\hat{k}_l\delta_{jk} - \hat{k}_i\hat{k}_k\delta_{jl} - \hat{k}_j\hat{k}_l\delta_{ik} - \hat{k}_j\hat{k}_k\delta_{il}\,, \tag{24}$$

where $\hat{k}_i = \frac{k_i}{k}$. We find that

$$G_{ijkl} = 2F_{ij}\delta_{kl} + 2F_{kl}\delta_{ij} - 2F_{ik}\delta_{jl} - 2F_{il}\delta_{jk} - 2F_{jl}\delta_{ik} - 2F_{jk}\delta_{il}$$
$$+2H_{ijkl} + 2D(x,x')(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \delta_{ij}\delta_{kl})\,. \tag{25}$$

Here $D(x,x')$, $F_{ij}(x,x')$ and $H_{ijkl}(x,x')$ are functions which are defined as follows:

$$D^n(x,x') = \frac{Re}{(2\pi)^{3+n}} \int \frac{d^{3+n}\mathbf{k}}{2\omega} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} e^{-i\omega(t-t')}\,, \tag{26}$$

$$F^n_{ij}(x,x') = \frac{Re}{(2\pi)^{3+n}} \partial_i\partial'_j \int \frac{d^{3+n}\mathbf{k}}{2\omega^3} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} e^{-i\omega(t-t')}\,, \tag{27}$$

and

$$H^n_{ijkl}(x,x') = \frac{Re}{(2\pi)^{3+n}} \partial_i\partial'_j\partial_k\partial'_l \int \frac{d^{3+n}\mathbf{k}}{2\omega^5} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} e^{-i\omega(t-t')}\,. \tag{28}$$

These functions are calculated in the Appendix. Let

$$R = |\mathbf{x}-\mathbf{x}'|, \qquad \Delta t = t - t'\,. \tag{29}$$

For $n = 2m+1$, an odd number of extra dimensions, the results are

$$D^{2m+1} = \begin{cases} \frac{(2m+1)!!}{2(2\pi)^{m+2}} \frac{1}{(R^2-\Delta t^2)^{m+\frac{3}{2}}}\,, & \text{for } R^2 > \Delta t^2\,, \\ 0 & \text{for } R^2 < \Delta t^2\,, \end{cases} \tag{30}$$

$$F^{2m+1}_{ij} = -\frac{1}{2(2\pi)^{m+2}}\, \partial_i\partial'_j \left[\frac{(2m)!!}{R^{2m}} S(0) \sum_{k=0}^{m} \frac{(2k+1)!!}{(2k)!!(2k+1)(2k-1)} \frac{R^{2k}}{(R^2-\Delta t^2)^k} \right]\,, \tag{31}$$

and

$$H^{2m+1}_{ijkl} = \frac{1}{2(2\pi)^{m+2}}\, \partial_i\partial'_j\partial_k\partial'_l \Bigg\{ \frac{(2m)!!}{R^{2m-2}} \Bigg[\frac{1}{2} Q(1)$$
$$+ \sum_{j=0}^{m-2} \frac{(m-j-1)(2j+1)!!}{4m(j+1)(2j)!!(2j+1)(2j-1)} \frac{R^{2j}}{(R^2-\Delta t^2)^j} S(0) \Bigg]\,. \tag{32}$$

In particular, for $n = 1$ ($d = 5$), we have

$$H^1_{ijkl} = 0 \tag{33}$$

Here

$$S(0) = \begin{cases} \frac{\sqrt{R^2-\Delta t^2}}{R^2} & \text{for } R^2 > \Delta t^2\,, \\ 0 & \text{for } R^2 < \Delta t^2\,, \end{cases} \tag{34}$$

and

$$Q(1) = \Big(\frac{1}{3} - \frac{\Delta t^2}{3R^2}\Big) S(0)\,. \tag{35}$$

For $n = 2m$, an even number of extra dimensions, we have

$$D^{2m} = \frac{2^m m!}{(2\pi)^{m+2}} \frac{1}{(R^2 - \Delta t^2)^{m+1}}\,, \tag{36}$$

$$\begin{aligned} F_{ij}^{2m} = \frac{1}{(2\pi)^{m+2}}\, \partial_i \partial_j' \Big\{ \frac{(2m-1)!!}{R^{2m}} \Big[1 - \frac{\Delta t}{4R} \ln\Big(\frac{R+\Delta t}{R-\Delta t}\Big)^2 \\ - \frac{1}{R^2} \sum_{k=2}^{m} \frac{2^{k-2}\Gamma(k-1)}{(2k-1)!!} \frac{R^{2k}}{(R^2-\Delta t^2)^{k-1}} \Big] \Big\} \qquad m \geq 1\,, \end{aligned} \tag{37}$$

and

$$\begin{aligned} H_{ijkl}^{2m} = \frac{1}{(2\pi)^{m+2}}\, \partial_i \partial_j' \partial_k \partial_l' \Big\{ \frac{(2m-1)!!}{R^{2m-4}} \Big[\frac{\Delta t}{24R^3} \Big(\frac{\Delta t^2}{R^2} - 1\Big) \ln\Big(\frac{R+\Delta t}{R-\Delta t}\Big)^2 - \frac{1}{18R^2} \\ - \frac{\Delta t^2}{6R^4} - \sum_{k=3}^{m} \frac{1}{(2k-1)(2k-3)} \left(\frac{1}{R^2} - \frac{\Delta t}{4R^3} \ln\Big(\frac{R+\Delta t}{R-\Delta t}\Big)^2 \right) \\ + \frac{1}{R^4} \sum_{j=2}^{m-2} \frac{(m-j-1) 2^{j-2}\Gamma(j-1)}{(2m-1)(2j+1)!!} \frac{R^{2j}}{(R^2-\Delta t^2)^{j-1}} \Big] \Big\} \qquad m \geq 2\,. \end{aligned} \tag{38}$$

For the case $n = 2$ ($d = 6$), we have for H

$$H_{ijkl}^{2} = \frac{1}{(2\pi)^3}\, \partial_i \partial_j' \partial_k \partial_l' \left[-\frac{1}{6} \ln(R^2 - \Delta t^2) - \frac{\Delta t^2}{6R^2} + \frac{\Delta t}{8R} \Big(\frac{\Delta t^2}{3R^2} - 1\Big) \ln\Big(\frac{R+\Delta t}{R-\Delta t}\Big)^2 \right]. \tag{39}$$

The case of $n = 0$ ($d = 4$) was given in Ref. [11].

III. PERIODIC COMPACTIFICATION SCENARIO

Let us now suppose that the extra n dimensions $z_1, ..., z_n$ are compactified with periodicity lengths $L_1, ..., L_n$, namely spatial points z_i and $z_i + L_i$ are identified. For simplicity, we shall assume in this paper that $L_1 = ... = L_n = L$. The effect of imposition of the periodic boundary conditions on the extra dimensions is to restrict the field modes to a discrete set

$$f_{\mathbf{k}} = (2\omega(2\pi)^3 L^n)^{-\frac{1}{2}} e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)}\,, \tag{40}$$

with

$$k_i = \frac{2\pi m_i}{L}, \quad i = 1, ..., n, \quad m_i = 0, \pm 1, \pm 2, \pm 3, \dots . \tag{41}$$

Let us denote the associated vacuum state by $|0_L\rangle$. In order to calculate the gravitational vacuum fluctuations due to compactification of extra dimensions, we need the renormalized graviton Hadamard function with respect to the vacuum state $|0_L\rangle$, $G^R_{\mu\nu\rho\sigma}(x, x')$, which is given by a multiple image sum of the corresponding Hadamard function for the Minkowski [3] vacuum, $G_{\mu\nu\rho\sigma}$:

$$G^R_{\mu\nu\rho\sigma}(t, z_i, t', z'_i) = \prod_{i=1}^{n} \sum_{m_i=-\infty}^{+\infty}{}' G_{\mu\nu\rho\sigma}(t, z_i, t', z'_i + m_i L) . \tag{42}$$

Here the prime on the summation indicates that the $m_i = 0$ term is excluded and the notation

$$(t, \vec{x}, z_1, .., z_n) \equiv (t, z_i) \tag{43}$$

has been adopted.

We are mainly concerned about how lightcone fluctuations arise in the usual uncompactified space as a result of compactification of extra dimensions. So we shall examine the case of a light ray propagating in one of the uncompactified dimensions. Take the direction to be along the x-axis in our four dimensional world, then the relevant graviton two-point function is G_{xxxx}, which can be expressed as

$$\begin{aligned} G_{xxxx}(t, \vec{x}, z_i, t', \vec{x}', z'_i) = 2\Big[& D(t, \vec{x}, z_i, t', \vec{x}', z'_i) - 2F_{xx}(t, \vec{x}, z_i, t', \vec{x}', z'_i) \\ & + H_{xxxx}(t, \vec{x}, z_i, t', \vec{x}', z'_i)\Big] . \end{aligned} \tag{44}$$

Assuming that the propagation goes from point $(a, 0, ..., 0)$ to point $(b, 0, ..., 0)$, we have

$$\begin{aligned} \langle\sigma_1^2\rangle &= \frac{1}{8}(b-a)^2 \int_a^b dx \int_a^b dx'\, G^R_{xxxx}(t, x, \mathbf{0}, t', x', \mathbf{0},), \\ &= \frac{1}{8}(b-a)^2 \int_a^b dx \int_a^b dx' \prod_{i=1}^{n} \sum_{m_i=-\infty}^{+\infty}{}' G_{xxxx}(t, x, \mathbf{0}, t', x', 0, 0, m_1 L, ..., m_i L) . \end{aligned} \tag{45}$$

With these results, we can in principle calculate lightcone fluctuations in spacetimes with arbitrary number of flat extra dimensions. In what follows, we first examine some particular cases, then make some observations for the general case.

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[3]By Minkowski we mean flat spacetime with all dimensions uncompactified

A. The five dimensional Kaluza-Klein model

One of the most intriguing and elegant ways of unifying gauge field theories with gravitation is the higher-dimensional generalizations of Kaluza-Klein theory. The original suggestion of Kaluza and Klein [1] was that electromagnetism and general relativity could be unified by starting with a five-dimensional version of the latter and then somehow arranging for the fifth dimension to become unobservable. This idea was further generalized to higher dimensions in attempts to unify non-Abelian gauge fields with gravitation and has been extensively studied in recent years in the context of supergravity and superstring theories. In the course of investigation of new features arising from the introduction of extra dimensions, the five-dimensional Kaluza-Klein theory has always been taken as a prototypical model to carry out explicit calculations to obtain a basic understanding. This is also our strategy here. In this subsection, we will derive the results which were summarized in Ref. [12].

1. Calculation of Δt

To begin, let us examine the influence of the compactification of the fifth (extra) dimension on the light propagation in our four dimensional world, by considering a light ray traveling along the x-direction from point a to point b, which is perpendicular to the direction of compactification. Define

$$\rho = x - x', \qquad b - a = r \tag{46}$$

and note the fact that the integration in Eq. (45) is to be carried out along the classical null geodesic on which $t - t' = \rho$. Then we obtain, after performing the differentiation in both D and F_{xx},

$$\begin{aligned} &G_{xxxx}(t, x, 0, 0, 0, t', x', 0, 0, mL') \\ &\quad = \frac{1}{4\pi^2}\Bigg[-\frac{\rho^6}{(\rho^2+m^2L^2)^3|mL|^3} - \frac{7\rho^4}{(\rho^2+m^2L^2)^3|mL|} \\ &\qquad + \frac{9\rho^2|mL|}{(\rho^2+m^2L^2)^3} - \frac{|mL|^3}{(\rho^2+mL^2)^3}\Bigg]. \end{aligned} \tag{47}$$

Thus, we have

$$\begin{aligned} \langle\sigma_1^2\rangle &= \frac{1}{8}r^2\int_a^b dx\int_a^b dx'\, G^R_{xxxx}(t, x, \mathbf{0},\, t', x', \mathbf{0},), \\ &= \frac{1}{8}r^2\int_a^b dx\int_a^b dx' \sum_{m=-\infty}^{+\infty}{}' G_{xxxx}(t, x, \mathbf{0},\, t', x', 0, 0, mL) \\ &= \frac{r^2}{32\pi^2 L}\sum_{m=1}^{\infty}\left[\frac{8}{m}\ln(1+\frac{\gamma^2}{m^2}) - \frac{2\gamma^2}{m^3} - \frac{8\gamma^2}{(\gamma^2+m^2)m}\right], \end{aligned} \tag{48}$$

where we have introduced a dimensionless parameter $\gamma = r/L$. We are interested here in the case in which $\gamma \gg 1$. It then follows that the summation is dominated, to the leading order, by the second term,

$$\langle \sigma_1^2 \rangle_R \approx -\frac{r^2}{16\pi^2 L} \sum_{n=1}^{\infty} \frac{\gamma^2}{n^3} = -\frac{\zeta(3) r^2 \gamma^2}{16\pi^2 L}\,, \tag{49}$$

where $\zeta(3)$ is the Riemann-zeta function. So, the mean deviation from the classical propagation time due to the lightcone fluctuations is

$$\Delta t \approx \sqrt{\frac{\zeta(3)}{16\pi^2 L}}\,\gamma = \sqrt{\frac{\zeta(3)}{16\pi^2 L}}\sqrt{32\pi G_5}\,\gamma = \sqrt{\frac{2\zeta(3) G_4}{\pi}}\gamma \approx \left(\frac{r}{L}\right) t_{pl}\,. \tag{50}$$

Here we have used the fact that $G_5 = G_4 L$, and $t_{pl} \approx 5.39 \times 10^{-44} s$ is the Planck time.

This result reveals that here the mean deviation in the arrival time increases linearly[4] with r and inversely with the size of the extra dimension. This effect is counter-intuitive in the sense that it grows as the size of the compactified dimension decreases. When r is of cosmological dimensions and L is sufficiently small, the effect is potentially observable.

2. Choice of Renormalization and Changes in L.

Note that here $\langle \sigma_1^2 \rangle_R$ has been renormalized to zero as $L \to \infty$. This is the most natural choice of renormalization, corresponding to the effect of the graviton fluctuations vanishing in the limit of noncompactified spacetime. This is analogous to setting a Casimir energy density to zero in the limit of infinite plate separation, However, if instead of renormalizing $\langle \sigma_1^2 \rangle$ against the vacuum with respect to $L \to \infty$, we take the manifold with compactified extra dimensions at some fixed sizes L to have $\langle \sigma_1^2 \rangle_R = 0$, then the lightcone fluctuations could seem to be renormalized away. The latter renormalization scheme is a logical possibility that we can consider, although it is unnatural as there seems to be nothing in the theory which picks out a particular finite value of L. In any case, if L is somehow allowed to vary, for example, as the universe evolves, then lightcone fluctuations would produce noticeable effects, as one could at most set $\langle \sigma_1^2 \rangle_R = 0$ at one point along the path of a light ray. It is particularly so, when we try to detect the spread in the arrival times of pulses from distant astrophysical sources, where we are looking back in time. Hence significant lightcone fluctuations may arise no matter what renormalization scheme one chooses if the size L is allowed to vary.

To get an understanding for the case of a changing L, let us assume that L changes with time at an extremely small rate, which is in fact required by experimental data on the time evolution of fundamental constants (see, for example, Refs. [20–23]). Then the evolution of L can be reasonably well approximated by a linear function of time: [5]

---

[4]In the usual four dimensional case with one compactified spatial dimension Δt grows linearly with the square root of r (see Ref. [11]).

[5]Strictly speaking, the functional dependence of L on time should be given by a yet-unknown underlying dynamical theory which governs how the extra dimensions evolve. However, the assumption of a linear dependence is good enough for our purpose of getting a basic idea about how the variation of L over time would affect our results.

$$L = L_i + \alpha t \tag{51}$$

Using this expression for L and redoing the calculations [6], one finds,

$$\begin{aligned}\langle\sigma_1^2\rangle_R &= \frac{r^2}{32\pi^2 L_f}\sum_{m=1}^{\infty}\left(\frac{L_f}{L_i}\right)^2\left[\frac{8}{m}\ln(1+\frac{\gamma^2}{m^2}) - \frac{2\gamma^2}{m^3} - \frac{8\gamma^2}{(\gamma^2+m^2)m}\right]\\ &\approx -\frac{\zeta(3)r^2\gamma^2}{16\pi^2 L_f}\left(\frac{L_f}{L_i}\right)^2,\end{aligned} \tag{52}$$

where L_i is the initial compactification size when the light ray is emitted, $L_f = L_i + \alpha r$ is the final size at the time of reception and $\gamma = r/L_f$. Here $\langle\sigma_1^2\rangle_R$ is renormalized with respect to $L \to \infty$. Therefore, one has for the mean time deviation from the classical propagation time

$$\Delta t = \frac{\sqrt{\langle\sigma_1^2\rangle_R}}{r} \approx \left(\frac{L_f}{L_i}\right)\left(\frac{r}{L_f}\right)t_{pl}\,. \tag{53}$$

Another possibility, as we have mentioned earlier, is to renormalize $\langle\sigma_1^2\rangle$ against that corresponding to the current size L_f, which implements the idea of setting the renormalized $\langle\sigma_1^2\rangle_R$ to be zero if L is fixed always or at least during the propagation of the light. This is accomplished by taking the difference between Eq. (52) and Eq. (49) with L being replaced by L_f to obtain

$$\langle\sigma_1^2\rangle_R \approx \left[1-\left(\frac{L_f}{L_i}\right)^2\right]\frac{\zeta(3)r^2\gamma^2}{16\pi^2 L_f}\,, \tag{54}$$

which leads to

$$\Delta t = \frac{\sqrt{\langle\sigma_1^2\rangle_R}}{r} \approx \sqrt{\left|1-\left(\frac{L_f}{L_i}\right)^2\right|}\left(\frac{r}{L_f}\right)t_{pl}\,. \tag{55}$$

Equations (53) and (55) demonstrate clearly that no matter what renormalization scheme is employed, one gets a nonzero lightcone fluctuation effect as long as L is changing. We want to point out here again that renormalizing $\langle\sigma_1^2\rangle$ with respect to $L \to \infty$ is far more natural than to a particular finite size L_f, since the latter seems to pick out a preferred size L_f without any convincing reason to do so.

3. Correlation of Pulses

The fluctuation in the flight time of pulses, Δt, can apply to the successive wave crests of a plane wave. This leads to a broadening of spectral lines from a distant source. Note,

---

[6] It is worth pointing out here that the graviton two-point function obtained simply by replacing the constant L with a changing one is not the exact two-point function that satisfies the appropriate equations, but it is a very good leading order approximation provided that $\alpha \ll 1$

however, that Δt is the expected variation in the arrival times of two successive crests only when the successive pulses are uncorrelated [17]. To determine the correlation, we need to compare $|\langle\sigma_1^2\rangle|$ and $|\langle\sigma_1\sigma_1'\rangle|$. The latter quantity is defined by

$$\langle\sigma_1\sigma_1'\rangle = \frac{1}{8}(\Delta r)^2 \int_{r_0}^{r_1} dr \int_{r_0}^{r_1} dr' \; n^\mu n^\nu n^\rho n^\sigma \; G^R_{\mu\nu\rho\sigma}(x,x')\,, \tag{56}$$

where the r-integration is taken along the mean path of the first pulse, and the r'-integration is taken along that of the second pulse. Here we will assume that $\Delta t \ll \Delta r$, so the slopes, v and v', of the two mean paths are approximately unity. Thus the two-point function in Eq. (56) will be assumed to be evaluated at $\rho = |\mathbf{x}-\mathbf{x}'| = |r-r'|$ and $\tau = |t-t'| = |r-r'-t_0|$. If $|\langle\sigma_1\sigma_1'\rangle| \ll |\langle\sigma_1^2\rangle|$, two pulses are uncorrelated, and otherwise they are correlated.

Let us now suppose the time separation of two pulses is T, and note that the relevant graviton two-point function can be expressed as

$$\begin{aligned}
&G_{xxxx}(t,x,0,0,0,t',x',0,0,nL)|_{t-t'=\rho-T} \\
&\quad= \frac{1}{4\pi^2}\Big[-\frac{1}{\beta^3} - \frac{8\rho^2}{\beta(\rho^2+n^2L^2)^2} + \frac{2}{\beta(\rho^2+n^2L^2)} \\
&\qquad + \frac{16\beta\rho^2}{(\rho^2+n^2L^2)^3} - \frac{4\beta}{(\rho^2+n^2L^2)^2} + \frac{2n^2L^2}{\beta^3(\rho^2+n^2L^2)}\Big] \\
&\quad\equiv -\frac{1}{4\pi^2}\frac{1}{\beta^3} + \frac{1}{4\pi^2} f(\rho,n)\,,
\end{aligned} \tag{57}$$

where,

$$\beta(n,\rho) = (n^2L^2 + 2\rho T - T^2)^{1/2}\,. \tag{58}$$

Utilizing the following integration relation

$$\int_a^b dx \int_a^b dx' f(x-x') = \int_0^r (r-\rho)[f(\rho)+f(-\rho)]\,d\rho\,, \tag{59}$$

one finds that

$$\langle\sigma_1\sigma_1'\rangle = A + B\,, \tag{60}$$

where

$$\begin{aligned}
A &= -\frac{r^2}{16\pi^2}\sum_{n=1}^{\infty}\int_0^r d\rho\,(r-\rho)\Big[\frac{1}{\beta(\rho,n)^3} + \frac{1}{\beta(-\rho,n)^3}\Big] \\
&= -\frac{r^2}{16\pi^2L^3}\sum_{n=1}^{\infty}\frac{L^4}{T^2}\Big(2\sqrt{n^2-\frac{T^2}{L^2}} - \sqrt{n^2+\frac{2rT-T^2}{L^2}} - \sqrt{n^2-\frac{2rT+T^2}{L^2}}\Big)\,,
\end{aligned} \tag{61}$$

and

$$
\begin{aligned}
B &= \frac{r^2}{16\pi^2}\sum_{n=1}^{\infty}\int_0^r d\rho\,(r-\rho)[f(\rho,n)+f(-\rho,n)] \\
&= \frac{r^2}{16\pi^2L^3}\sum_{n=1}^{\infty} 2\Big[-\frac{2\sqrt{n^2L^2-T^2}}{n^2L^2}+\frac{\beta(r,n)+\beta(-r,n)}{(n^2L^2+r^2)}+\frac{2}{nL}\ln\Big(\frac{nL+\sqrt{n^2L^2-T^2}}{nL-\sqrt{n^2L^2-T^2}}\Big] \\
&\quad -\frac{1}{nL}\ln\Big(\frac{n^2L^2+rT+nL\beta(r,n)}{n^2L^2+rT-nL\beta(r,n)}\Big)-\frac{1}{nL}\ln\Big(\frac{n^2L^2-rT+nL\beta(-r,n)}{n^2L^2-rT-nL\beta(-r,n)}\Big)\Big)\,.
\end{aligned}
\tag{62}
$$

A few things are to be noticed here: (1) We need to drop those terms in A and B when the square root is imaginary. (2) It can be shown that the above expression for $\langle\sigma_1\sigma_1'\rangle$ reduces to $\langle\sigma_1^2\rangle$ when $T=0$, as it should. (3) The asymptotic behaviors of the summands when $n\to\infty$, are $\sim\frac{2}{n^3}$ for A and $\sim\frac{2}{n^5}$ for B, hence both A and B converge. (4) If $r\gg T, L$, then A dominates over B, since the leading order of the summand in A is $\sqrt{r}$ while that in B is a constant independent of r as $r\to\infty$. Thus $\langle\sigma_1\sigma_1'\rangle\approx A$.

To proceed, let us now assume that $r\gg T$ and $r\gg L$, then

$$
p\equiv\sqrt{\frac{2rT+T^2}{L^2}}\approx\sqrt{\frac{2rT}{L^2}}\gg 1 \tag{63}
$$

is a huge number. Thus for the third term in Eq. (61), the sum should only start from $n=p$. We can now split the summation into two parts, i.e. terms with $n\le p$ and those with $n>p$. Using the asymptotic form of the summand for the part with $n>p$ and defining $m=[T/L]$, where [ ] denotes the integer part, one has

$$
\langle\sigma_1\sigma_1'\rangle\approx-\frac{r^2}{16\pi^2L^3}\Big(\frac{2L^4}{T^2}\sum_{n=m}^{p}\sqrt{n^2-m^2}-\frac{L^4}{T^2}\sum_{n=1}^{p}\sqrt{n^2+\frac{2rT-T^2}{L^2}}+\sum_{p}^{\infty}\frac{2r^2}{n^3}\,.\Big) \tag{64}
$$

Hence, it follows that

$$
|\langle\sigma_1\sigma_1'\rangle|<\frac{r^2}{16\pi^2L^3}\Big(\frac{2L^4}{T^2}\sum_{n=1}^{p}n+\frac{L^4}{T^2}\sum_{n=1}^{p}\sqrt{n^2+\frac{2rT-T^2}{L^2}}+\sum_{p}^{\infty}\frac{2r^2}{n^3}\,.\Big) \tag{65}
$$

Let us now evaluate the above expression term by term. One has, keeping in mind that $p\gg 1$, that

$$
\begin{aligned}
\sum_{n=1}^{p}\sqrt{n^2+\frac{2rT-T^2}{L^2}} &\lesssim \sum_{n=1}^{p}\sqrt{n^2+p^2}=p\sum_{n=1}^{p}\sqrt{n^2/p^2+1} \\
&\approx p^2\int_{1/p}^{1}\sqrt{x^2+1}\,dx\approx\frac{1}{2}[\sqrt{2}+\ln(\sqrt{2}+1)]\,p^2\,,
\end{aligned}
\tag{66}
$$

and

$$
\sum_{p}^{\infty}\frac{r^2}{n^3}=-\frac{r^2}{2}\Psi(2,p)\sim\frac{r^2}{2}\frac{1}{p^2}=\frac{1}{4}\frac{L}{r}\frac{L}{T}r^2\,. \tag{67}
$$

Here we have used Eq. (63) and the asymptotic expansion for $\Psi(2,x)$

$$\Psi(2,x) \approx -\frac{1}{x^2} - \frac{1}{x^3} - \frac{1}{2x^4} + O(1/x^6)\,, \tag{68}$$

where function $\Psi(n,x)$ is defined as

$$\Psi(n,x) = \frac{d^n \psi(x)}{dx^n}\,, \qquad \psi(x) = \frac{d}{dx}\ln\Gamma(x)\,, \tag{69}$$

Noting that for $p \gg 1$,

$$\sum_{n=1}^{p} n \sim \frac{1}{2}p^2\,, \tag{70}$$

and letting

$$A = 1 + \frac{1}{2}[\sqrt{2} + \ln(\sqrt{2}+1)]\,, \tag{71}$$

we finally find

$$|\langle\sigma_1\sigma_1'\rangle| < \frac{r^2}{16\pi^2 L^3}\left(A\frac{L^4}{T^2}p^2 + \frac{r^2}{2}\frac{1}{p^2}\right) = \frac{r^2}{16\pi^2 L^3}\left(2A + \frac{1}{2}\right)\frac{L}{r}\frac{L}{T}\,r^2\,. \tag{72}$$

Compare this result with

$$|\langle\sigma_1^2\rangle_R| \approx \frac{r^2}{16\pi^2 L^3}\zeta(3)\,r^2\,. \tag{73}$$

It is seen that two successive pulses separated by T in time are uncorrelated ($|\langle\sigma_1\sigma_1'\rangle| \ll |\langle\sigma_1^2\rangle_R|$) provided that

$$r \gg \frac{L^2}{T}\,, \tag{74}$$

or equivalently

$$T \gg \frac{L^2}{r} = \frac{L}{r}\,L\,. \tag{75}$$

However, if $r \ll L$, one can show, by series expanding both A and B, that

$$|\langle\sigma_1\sigma_1'\rangle| \approx \frac{r^2}{16\pi^2 L^3}\sum_{n=1}^{\infty}\frac{1}{(n^2 - \frac{T^2}{L^2})^{3/2}}\,. \tag{76}$$

Clearly, in this case, $|\langle\sigma_1\sigma_1'\rangle| \ll |\langle\sigma_1^2\rangle_R|$, if $T \gg L$, and $|\langle\sigma_1\sigma_1'\rangle| \approx |\langle\sigma_1^2\rangle_R|$, when $T < L$.

A few comments are now in order about the physical picture behind our correlation results. It is natural to expect from the configuration that the dominant contributions to the light cone fluctuation come from the graviton modes with wavelengths of the order of $\sim L$. In other words, the lightcone fluctuates on a typical time scale of $\sim 1/L$. If the travel distance, r, is less than L, successive pulses are uncorrelated only when their time separation is greater than the typical fluctuation time scale. Otherwise they are correlated because the quantum gravitational vacuum fluctuations are not significant enough in the interval between the pulses. However, if $r \gg L^2/T$, then successive pulses are in general uncorrelated. Thus the correlation time for large r is of order L^2/r, which is much smaller than the compactification scale L. We can understand this result as arising from the loss of correlation as the pulses propagate over an increasing distance.

Suppose that the experimental fractional resolution for a particular spectral line of period T is Γ. Then we must have

$$\frac{\Delta t}{T} \leq \Gamma \,, \tag{77}$$

which, with Eq. (50), leads to a bound on L of

$$L \geq \frac{r\, t_{pl}}{\Gamma T} \,, \tag{78}$$

assuming L does not change over time. However, this bound can be trusted only when two successive wave crests are uncorrelated, when Δt is the expected variation in their arrival times. The most conservative constraint from this requirement is that L is smaller than the wavelength of the spectral line, T. Namely,

$$\frac{r\, t_{pl}}{\Gamma T} \leq T \,, \tag{79}$$

yielding a restriction on the range of spectral lines that we should use to get the lower bound

$$T \geq \sqrt{\frac{r\, t_{pl}}{\Gamma}} \,. \tag{80}$$

If Eq. (80) is approximately an equality, then Eq. (78) becomes

$$L \geq \left(\frac{r t_{pl}}{\Gamma}\right)^{\frac{1}{2}} = \left(\frac{r l_{pl}}{\Gamma}\right)^{\frac{1}{2}} \,. \tag{81}$$

Obviously, the optimal lower bound would be deduced from the spectral lines of distant galaxies, possibly of cosmological distance, with the highest observed spectral resolution. For astrophysical sources of cosmological distance, spectral lines which satisfy Eq. (80) will have wavelengths $\gtrsim$ 1mm assuming a resolution of $\Gamma \approx 10^{-3}$ and a cosmological travel distance. The detection of CO(1$\longrightarrow$0) line emission at 2.6 mm from luminous infrared galaxies and quasars [24] provides the type of data needed to get a bound. According to Ref. [24], the observed CO line resolution for the infrared quasar IRAS 07598+6508 , which is at a distance of 596 Mpc assuming $H_0 = 75$ km s^{-1} Mpc^{-1}, is about $\Gamma \approx 10^{-3}$, leading to

$$L \gtrsim 10^{-1}\,\text{mm} \,. \tag{82}$$

Here, the size of the extra dimension has to be macroscopically large in order not to contradict the astrophysical observation. The lower bound given here is within the sensitivity of the recently proposed experiments for possible deviations from Newtonian gravity [25].

Note, however, this bound could be pushed to an even higher value if we do not require $L < T$. This is in fact legitimate, since for astrophysical sources one usually has both $r \gg L$ and $r \gg T$ satisfied, and the condition for two successive pulses to be uncorrelated is really

$$r > \frac{L^2}{T} \,. \tag{83}$$

Hence, we can use Eq. (78) to get a bound from experimental data as long as the resulting bound, L, obeys Eq. (83),

$$L < \sqrt{rT} \equiv L_0 \,. \tag{84}$$

Thus, a stronger bound can be achieved, if we can find astrophysical sources of cosmological distance with observed spectra of much smaller wavelengths provided this condition is satisfied. The observation of γ rays from astrophysical sources, such as gamma-ray bursters (GRBs) [26,27], provides such an opportunity. The use of these sources as probes of possible quantum gravity effects has been explored by a number of authors [28–32] recently. Below we will select some of these γ ray sources to calculate both the lower bound L_b from Eq. (78) and L_0 from Eq. (84). To be conservative, we shall assume a resolution of the order of unity for all the gamma rays we are going to consider, since we should at least have this resolution before we can talk with confidence about the observed energies (or frequencies) of the gamma rays.

In the following Table, we list the source names, the observed frequencies ν, the source distances D, and the calculated L_b and L_0.

Table: Bounds on L from GRB sources

Source	D(Mpc)	$\nu(Hz)$	$L_b(mm)$	$L_0(mm)$
GRB 930229 [33]	791	4.8×10^{19}	6.1×10^4	1.2×10^{10}
GRB 940217 [34]	385	4.3×10^{24}	2.6×10^9	2.8×10^7
GRB 930131 [35]	260	1.1×10^{23}	4.6×10^7	1.5×10^8
Mrk 421 [30]	112	4.8×10^{26}	8.7×10^{10}	1.4×10^6
GRB 980703 [29]	1592	1.2×10^{20}	3.1×10^5	1.0×10^{10}
GRB 980425 [36]	40	5.4×10^{20}	3.4×10^4	8.3×10^8
GRB 990123 [37]	2400	3.0×10^{20}	1.2×10^6	8.5×10^9

From this Table, one can find that the largest lower bound for L comes from GRB930131 and GRB990123, which is

$$L \gtrsim 10^7 \text{mm} \,. \tag{85}$$

For some sources, Mrk 421, for instance, one seems to get a much larger bound, which however can not be trusted, because the correlation condition $L_b < L_0$ is violated. The physical reason is that the frequencies for the gamma rays are so high that even travel over a cosmological distance does not wash out the correlation between successive wave crests.

In principle, the above results only apply to the case where L is fixed. When L changes as the universe evolves, one should use Eq. (53), or Eq. (55), for the mean time deviation, Δt. In this case, we can set either a bound on L or a bound on the rate of change of L, if we can constrain either the rate of change of the size of extra dimensions over time or the present size L from other considerations. Let L_f be the size of the extra dimension at the present time and L_i be that at the time of primordial nucleosynthesis, and write

$$\frac{L_f}{L_i} = 1 + \delta \,. \tag{86}$$

According to Refs. [20,21], the observational limits, obtained by investigating the effects on the primordial nucleosynthesis of ^{4}He as a consequence of the time variation of fundamental constants such as the electroweak, strong, and gravitational coupling constants, imply that $\delta \lesssim 0.01$. However, stronger limits on δ, which may be less reliable, arise from a detailed study of the events which took place 1.8×10^9yr ago on the current site of an open-pit uranium mine at Oklo in the West African Republic of Gabon [22]. This site gave rise to a natural nuclear reactor when it went critical for a period about 1.8×10^9yr ago. The Oklo samples constrain the rate of change of extra spatial dimensions to satisfy the limits [23,21]

$$\left|\frac{\dot{L}}{L}\right| \leq 1.9 \times 10^{-19}\mathrm{yr}^{-1}\,, \tag{87}$$

which translates to $\delta \lesssim 10^{-10}$. In either case, $\delta \ll 1$. Thus all the results obtained so far for a fixed L hold for a changing L if renormalization with respect to $L \to \infty$ is adopted (refer to Eq. (53)). However if one chooses the renormalization with respect to the current size L_f, then combination of Eq. (77) and Eq. (55) gives rise to

$$L \geq \sqrt{\delta^2 + 2\delta}\left(\frac{rt_{pl}}{T\Gamma}\right) \approx \sqrt{\delta}\left(\frac{rt_{pl}}{T\Gamma}\right). \tag{88}$$

As a result, the following considerably smaller lower bounds can be deduced from Eq. (88) using the CO line data

$$\begin{aligned} L &\gtrsim 10^{-2}\,\mathrm{mm}\,, \quad \text{for} \quad \delta = 0.01\,, \\ L &\gtrsim 10^{-6}\,\mathrm{mm}\,, \quad \text{for} \quad \delta = 10^{-10}\,, \end{aligned} \tag{89}$$

and

$$\begin{aligned} L &\gtrsim 10^{6}\,\mathrm{mm}\,, \quad \text{for} \quad \delta = 0.01\,, \\ L &\gtrsim 10^{2}\,\mathrm{mm}\,, \quad \text{for} \quad \delta = 10^{-10}\,, \end{aligned} \tag{90}$$

using GRB data.

On the other hand, one can also use our result to place a bound on the rate of change of L if the present size of L can be fixed by other considerations. In this respect, it is well-known that the five-dimensional Kaluza-Klein theory provides an explanation of the quantization of electric charge, in the sense that all electric charges are multiples of the elementary charge

$$e = \frac{4\sqrt{\pi G}}{L}\,. \tag{91}$$

The corresponding fine structure constant is then

$$\alpha_f = \frac{4G}{L^2}\,. \tag{92}$$

Setting α_f to its present value, 1/137, we get an estimate of the size of the fifth dimension, $L \sim 10^{-31}$ cm, in the original Kaluza-Klein model. For the case of renormalization with respect to $L \to \infty$, we can see from Eq. (77), Eq. (53) and Eq. (86) that we must have $\delta < 1$, which is much weaker than the existing bounds on the change rate from primordial nucleosynthesis and the Oklo samples. To discuss the case of renormalization with respect to L_f, the current size, let us write Eq. (88) as

$$\delta \leq \left(\frac{LT\Gamma}{rt_{pl}}\right)^2 . \tag{93}$$

Thus, using $L \sim 10^{-31}$ cm and the same CO line data as before, we find the following limit for change of the extra dimension in the Kaluza-Klein model

$$\delta \leq 10^{-58} , \tag{94}$$

or the rate of change by dividing δ by the travel time, which is $\sim 10^9$ yr

$$\left|\frac{\dot{L}}{L}\right| \leq 10^{-67}\mathrm{yr}^{-1} . \tag{95}$$

This is much stronger even than the strongest bound arising from the observational limits on the time evolution of fundamental constants.

To conclude, we have demonstrated, in the case of one extra dimension, that the large quantum lightcone fluctuations due to the compactification of the extra dimension require either the size of the extra dimension to be macroscopically large or rate of change of the extra dimension to be extremely small. This result seems to rule out the five dimensional Kaluza-Klein theory, or at the very least, place strong limits on the rate of change of the extra dimension. We must point out that the rate of growth of Δt with r depends crucially on the number of spatial dimensions. In four dimensions, $\Delta t \propto \sqrt{r}$, while in five dimensions $\Delta t \propto r$. One expects that in larger number of dimensions, there will be an effect of compactification, but its details need to be determined by explicit calculations for particular models. This is the topic for the next section.

B. Higher dimensional models

There was no real reason to extend the Kaluza-Klein idea beyond five dimensions until the emergence of non-abelian gauge field theories which have had a profound impact on theoretical physics since their invention by Yang and Mills in 1954. It was suggested by DeWitt [38] as early as in 1963 that a unification of Yang-Mills theories and gravitation could be achieved in a higher dimensional Kaluza-Klein framework. Nowadays, the possibility of unifying all the known interactions in nature in higher dimensional spacetimes has been actively pursued in the context of eleven dimensional supergravity and ten dimensional superstring theories. A necessary ingredient of all these higher dimensional models is the compactification of the extra dimensions to a very small size so as to leave the ordinary four-dimensional "large" world.

In this section, we examine, case by case, the lightcone fluctuations arising from quantum gravitational vacuum fluctuations induced by the periodic compactification of extra, flat spatial dimensions in higher dimensional models up through 11 dimensions and make a conjecture about arbitrary dimensions.

a. The case with $n = 2$ This is the 6 dimensional spacetime with 2 extra dimensions. We obtain, after performing the differentiation in both F_{xx} and H_{xxxx} (See Eqs (37) and (38)).

$$G_{xxxx}(t, x, \mathbf{0},\, t', x', 0, 0, m_1 L, m_2 L) \equiv A_2(\rho) + B_2(\rho) \tag{96}$$

with

$$A_2(\rho) = \frac{1}{4\pi^3}\left[-\frac{54\rho^6}{(\rho^2+\alpha_2^2)^5} + \frac{102\alpha_2^2\rho^4}{(\rho^2+\alpha_2^2)^5} - \frac{3\alpha_2^4\rho^2}{2(\rho^2+\alpha_2^2)^5}\right], \tag{97}$$

and

$$\begin{aligned} B_2(\rho) = \frac{1}{4\pi^3}&\left[\frac{6\rho^7}{(\rho^2+\alpha_2^2)^{\frac{11}{2}}} - \frac{27\alpha_2^2\rho^5}{(\rho^2+\alpha_2^2)^{\frac{11}{2}}} + \frac{27\alpha_2^4\rho^3}{4(\rho^2+\alpha_2^2)^{\frac{11}{2}}} + \frac{3\alpha_2^6\rho}{8(\rho^2+\alpha_2^2)^{\frac{11}{2}}}\right]\times \\ &\ln\left(\frac{\sqrt{\rho^2+\alpha_2^2}+\rho}{\sqrt{\rho^2+\alpha_2^2}-\rho}\right)^2, \end{aligned} \tag{98}$$

where we have introduced a parameter

$$\alpha_n^2 = L^2\sum_{i=1}^{n} m_i^2\,. \tag{99}$$

Upon noting that G_{xxxx} is an even function of ρ, it follows that,

$$\int_a^b dx \int_a^b dx'\, G^R_{xxxx}(t, x, \mathbf{0},\, t', x', \mathbf{0}) = \prod_{i=1}^{2}\sum_{m_i=-\infty}^{+\infty}\int_0^r 2(r-\rho)(A_2+B_2)d\rho\,. \tag{100}$$

Performing the integration (integrate by parts for those terms involving logarithmic functions), we find

$$\begin{aligned} &\prod_{i=1}^{2}\sum_{m_i=-\infty}^{+\infty}\int_0^r 2(r-\rho)(A_2+B_2)d\rho \\ &\quad= \frac{1}{2\pi^3 L^2}\sum_{m_1=1}^{\infty}{}'\sum_{m_2=1}^{\infty}{}'\left[\frac{\gamma^2(m_1^2+m_2^2)}{(\gamma^2+m_1^2+m_2^2)^3} + \frac{10\gamma^4}{3(\gamma^2+m_1^2+m_2^2)^3}\right. \\ &\qquad\qquad -\frac{8\gamma^6}{3(m_1^2+m_2^2)(\gamma^2+m_1^2+m_2^2)^3} - \frac{8\gamma^5+4(m_1+m_2^2)\gamma^3+(m_1^2+m_2^2)^2\gamma}{2(m_1^2+m_2^2+\gamma^2)^{7/2}} \\ &\qquad\qquad\quad \left.\times\ln\left(\frac{\sqrt{m_1^2+m_2^2+\gamma^2}+\gamma}{\sqrt{m_1^2+m_2^2+\gamma^2}-\gamma}\right)\right], \end{aligned} \tag{101}$$

where $\gamma = r/L$ is a dimensionless parameter. The above double summation is by no means easy to evaluate. However, we are interested in the case in which the travel distance r is much greater than the size of the extra dimensions L, i.e., $\gamma \gg 1$. It then follows that the summation is dominated, to the leading order, by

$$-\frac{1}{2\pi^3 L^2}\sum_{m_1=1}^{\infty}{}' \sum_{m_2=1}^{\infty}{}' \frac{8\gamma^6}{3(m_1^2+m_2^2)(\gamma^2+m_1^2+m_2^2)^3}\,, \tag{102}$$

which can be approximated by the following integral when $\gamma \gg 1$

$$-\frac{4}{3\pi^3 L^2}\int_{1/\gamma}^{\infty} dx_1 \int_{1/\gamma}^{\infty} dx_2 \,\frac{1}{(x_1^2+x_2^2)(x_1^2+x_2^2+1)^3} \approx \frac{4}{3\pi^3 L^2}\ln(\gamma)\,, \qquad \text{as } \gamma \to \infty. \tag{103}$$

An easy way to see the above behavior is to note that the contribution to the integral is dominated by the region around $(1/\gamma, 1/\gamma)$ since the integrand dies away very quickly as x_1 and x_2 increase, and then change to polar coordinates to evaluate the integral while using the fact that around $(1/\gamma, 1/\gamma)$ the integrand is approximated by $1/(x_1^2+x_2^2)$. Therefore one finds that

$$\langle \sigma_1^2 \rangle_R \approx \frac{r^2}{6\pi^3 L^2}\ln(\gamma)\,, \tag{104}$$

and the mean deviation from the classical propagation time due to the lightcone fluctuations

$$\Delta t = \frac{\sqrt{\langle \sigma_1^2\rangle_R}}{r} \approx \sqrt{\frac{32\pi G_6}{6\pi^3 L^2}}\sqrt{\ln(\gamma)} = \sqrt{\frac{16 G_4}{3\pi^2}}\sqrt{\ln(\gamma)} \approx \frac{4 t_{pl}}{\sqrt{3}\pi}\sqrt{\ln(r/L)}\,. \tag{105}$$

Here we have used the fact that $G_{4+n} = G_4 L^n$. Note that here the Δt dependence on γ is square root of a logarithmic function which is quite different from linear dependence in the case with $n = 5$ discussed above. So, the mean time deviation here grows much more slowly as the travel distance r increases or the compactification scale decreases. As we shall see later, this seems to a general feature for $n > 5$.

b. The case with n=3 This is the 7 dimensional spacetime with 3 extra dimensions. The relevant two-point function can be shown to be

$$\begin{aligned} &G_{xxxx}(t,x,\mathbf{0},\, t',x',0,0,m_1L,...,m_3L) \\ &= \frac{1}{8\pi^3}\left[\frac{24\rho^8}{\alpha_3(\rho^2+\alpha_3^2)^6} - \frac{336\alpha_3\rho^6}{(\rho^2+\alpha_3^2)^6} + \frac{280\alpha_3^3\rho^4}{2(\rho^2+\alpha_2^2)^6}\right]\,, \end{aligned} \tag{106}$$

which leads to

$$\begin{aligned} &\int_a^b dx \int_a^b dx'\, G^R_{xxxx}(t,x,\mathbf{0},\, t',x',\mathbf{0}) \\ &\quad = -\frac{1}{4\pi^3 L^3}\prod_{i=1}^{3}\sum_{m_1=1}^{\infty}{}' \Bigg(\frac{8\gamma^8}{3(\sum_{i=1}^3 m_i^2)^{\frac{3}{2}}(\gamma^2+\sum_{i=1}^3 m_i^2)^4} \\ &\qquad + \frac{56\gamma^6}{3(\sum_{i=1}^3 m_i^2)^{\frac{1}{2}}(\gamma^2+\sum_{i=1}^3 m_i^2)^4}\Bigg)\,. \end{aligned} \tag{107}$$

The triple summation is dominated, to the leading order, by the first term when $\gamma \gg 1$, which again can be approximated by integration to be

$$-\frac{2}{3\pi^3 L^3}\int_{1/\gamma}^{\infty} dx_1 \int_{1/\gamma}^{\infty} dx_2 \int_{1/\gamma}^{\infty} dx_3 \frac{1}{(\sum_{i=1}^3 x_i^2)^{\frac{3}{2}}(\sum_{i=1}^3 +1)^4} \approx \frac{2}{3\pi^3 L^3}\ln(\gamma)\,, \tag{108}$$

as $\gamma \to \infty$. Thus the mean deviation from the classical propagation time due to the lightcone fluctuations

$$\Delta t = \frac{\sqrt{\langle \sigma_1^2 \rangle_R}}{r} \approx \sqrt{\frac{32\pi G_7}{12\pi^3 L^3}}\,\sqrt{\ln(\gamma)} = \sqrt{\frac{8G_4}{3\pi^2}}\sqrt{\ln(\gamma)} \approx \frac{2\sqrt{2}t_{pl}}{\sqrt{3}\pi}\,\sqrt{\ln(\gamma)}\,. \tag{109}$$

c. The case with n=4 This is the 8 dimensional spacetime with 4 extra dimensions. The relevant two-point function is

$$G_{xxxx}(t,x,\mathbf{0},\,t',x',0,0,m_1L,...,m_4L) \equiv A_4(\rho) + B_4(\rho) \tag{110}$$

with

$$A_4 = \frac{1}{8\pi^4}\Bigg[-\frac{912\rho^6}{(\rho^2+\alpha_4^2)^6} + \frac{902\alpha_4^2\rho^4}{(\rho^2+\alpha_4^2)^6} + \frac{201\alpha_4^4\rho^2}{2(\rho^2+\alpha_4^2)^6} + \frac{24\rho^8}{\alpha_4^2(\rho^2+\alpha_4^2)^6} - \frac{12\alpha_4^6}{(\rho^2+\alpha_4^2)^6}\Bigg], \tag{111}$$

and

$$B_4 = \frac{1}{8\pi^4}\left[\frac{240\rho^7}{(\rho^2+\alpha_4^2)^{\frac{13}{2}}} - \frac{540\alpha_4^2\rho^5}{(\rho^2+\alpha_4^2)^{\frac{13}{2}}} + \frac{90\alpha_4^4\rho^3}{(\rho^2+\alpha_4^2)^{\frac{13}{2}}} + \frac{15\alpha_4^6\rho}{4(\rho^2+\alpha_2^2)^{\frac{11}{2}}}\right] \times \ln\left(\frac{\sqrt{\rho^2+\alpha_4^2}+\rho}{\sqrt{\rho^2+\alpha_4^2}-\rho}\right)^2 . \tag{112}$$

One finds after carrying out the integration

$$\begin{aligned}
&\int_a^b dx \int_a^b dx'\, G^R_{xxxx}(t,x,\mathbf{0},\,t',x',\mathbf{0}) \\
&= \frac{1}{4\pi^4 L^4}\prod_{i=1}^{4}\sum_{m_i=1}^{\infty}{}'\Bigg[-\frac{64\gamma^6}{3(\sum_{i=1}^4\, m_i^2)(\gamma^2+\sum_{i=1}^4\, m_i^2)^4} \\
&\quad -\frac{10\gamma^8}{3(\sum_{i=1}^4\, m_i^2)^2(\gamma^2+\sum_{i=1}^4\, m_i^2)^4} + \frac{32\gamma^4}{(\gamma^2+\sum_{i=1}^4\, m_i^2)^4} + \frac{15\gamma^2(\sum_{i=1}^4\, m_i^2)}{(\gamma^2+\sum_{i=1}^4\, m_i^2)^4} \\
&\quad -\frac{48\gamma^5+16(\sum_{i=1}^4\, m_i^2)\gamma^3+3(\sum_{i=1}^4\, m_i^2)^2\gamma}{2(\sum_{i=1}^4\, m_i^2+\gamma^2)^{9/2}}\ln\left(\frac{\sqrt{\sum_{i=1}^4\, m_i^2+\gamma^2}+\gamma}{\sqrt{\sum_{i=1}^4\, m_i^2+\gamma^2}-\gamma}\right)\Bigg].
\end{aligned} \tag{113}$$

The summation is seen to be dominated by the second term when $\gamma \gg 1$ and in that case the summation turns out to be approximated by an integral as

$$-\frac{5}{6\pi^4 L^4}\prod_{i=1}^{4}\int_{1/\gamma}^{\infty} dx_i\,\frac{1}{(\sum_{i=1}^5 x_i^2)^2(\sum_{i=1}^5 x_i^2+1)^4} \approx \frac{5}{6\pi^4 L^4}\ln(\gamma)\,. \tag{114}$$

Therefore, one obtains

$$\langle \sigma_1^2 \rangle_R \approx \frac{5r^2}{48\pi^4 L^4}\ln(\gamma)\,, \tag{115}$$

and

$$\Delta t = \frac{\sqrt{\langle \sigma_1^2 \rangle_R}}{r} \approx \frac{\sqrt{10}t_{pl}}{\sqrt{3}\pi^{\frac{3}{2}}}\,\sqrt{\ln(\gamma)}\,. \tag{116}$$

d. The case with n=5 This is the 9 dimensional spacetime with 5 extra dimensions. One finds in this case that

$$
\begin{aligned}
&G_{xxxx}(t, x, \mathbf{0},\, t', x', 0, 0, m_1 L, ..., m_2 L) \\
&\quad = \frac{1}{16\pi^3}\Bigg[\frac{24\rho^{10}}{\alpha_5^3(\rho^2+\alpha_5^2)^7}\frac{648\rho^8}{\alpha_5(\rho^2+\alpha_5^2)^7} - \frac{4536\alpha_5\rho^6}{(\rho^2+\alpha_5^2)^7} + \frac{2520\alpha_5^3\rho^4}{2(\rho^2+\alpha_5^3)^7}\Bigg],
\end{aligned}
\tag{117}
$$

and

$$
\begin{aligned}
&\int_a^b dx \int_a^b dx'\, G^{(1)R}_{xxxx}(t, x, 0, 0,\, t', x', 0, m_i L) \\
&= -\frac{1}{8\pi^3 L^5}\prod_{i=1}^{5}\sum_{m_i=1}^{\infty}{}' \Bigg(\frac{8\gamma^{10}}{(\sum_{i=1}^5 m_i^2)^{\frac{5}{2}}(\gamma^2+\sum_{i=1}^5 m_i^2)^5} + \frac{48\gamma^8}{(\sum_{i=1}^5 m_i^2)^{\frac{3}{2}}(\gamma^2+\sum_{i=1}^5 m_i^2)^5} \\
&\quad + \frac{168\gamma^6}{(\sum_{i=1}^5 m_i^2)^{\frac{1}{2}}(\gamma^2+\sum_{i=1}^5 m_i^2)^5}\Bigg).
\end{aligned}
\tag{118}
$$

The summation, to the leading order, can be approximated by integration

$$
-\frac{1}{\pi^4 L^5}\prod_{i=1}^{5}\int_{1/\gamma}^{\infty} dx_i\, \frac{1}{(\sum_{i=1}^5 x_i^2)^{\frac{5}{2}}(\sum_{i=1}^5 x_i^2+1)^5} \approx \frac{1}{\pi^4 L^5}\ln(\gamma)\,,
\tag{119}
$$

as $\gamma \to \infty$. Thus the mean deviation from the classical propagation time due to the lightcone fluctuations

$$
\Delta t = \frac{\sqrt{\langle \sigma_1^2\rangle_R}}{r} \approx \frac{2t_{pl}}{\pi^{\frac{3}{2}}}\sqrt{\ln(\gamma)}\,.
\tag{120}
$$

e. The case with n=6 This is the 10 dimensional spacetime with 4 extra dimensions as motivated by superstring theory. The relevant two-point function is given by

$$
G_{xxxx}(t, x, \mathbf{0},\, t', x', 0, 0, m_1 L, ..., m_6 L) \equiv A_6(\rho) + B_6(\rho)\,,
\tag{121}
$$

where

$$
\begin{aligned}
A_6 = \frac{1}{16\pi^5}\Bigg[&-\frac{15102\rho^6}{(\rho^2+\alpha_6^2)^7} + \frac{9102\alpha_6^2\rho^4}{(\rho^2+\alpha_6^2)^7} + \frac{4575\alpha_6^4\rho^2}{2(\rho^2+\alpha_6^2)^7} \\
&+\frac{816\rho^8}{\alpha_6^2(\rho^2+\alpha_6^2)^7} - \frac{180\alpha_6^6}{(\rho^2+\alpha_6^2)^7} + \frac{48\rho^{10}}{\alpha_6^4(\rho^2+\alpha_6^2)^7}\Bigg],
\end{aligned}
\tag{122}
$$

and

$$
\begin{aligned}
B_6 = \frac{1}{16\pi^5}&\left[\frac{4200\rho^7}{(\rho^2+\alpha_6^2)^{\frac{15}{2}}} - \frac{6300\alpha_6^2\rho^5}{(\rho^2+\alpha_6^2)^{\frac{15}{2}}} + \frac{1575\alpha_6^4\rho^3}{2(\rho^2+\alpha_6^2)^{\frac{15}{2}}} + \frac{105\alpha_6^6\rho}{4(\rho^2+\alpha_6^2)^{\frac{11}{2}}}\right]\times \\
&\ln\left(\frac{\sqrt{\rho^2+\alpha_6^2}+\rho}{\sqrt{\rho^2+\alpha_6^2}-\rho}\right)^2.
\end{aligned}
\tag{123}
$$

One finds after carrying out the integration

$$
\begin{aligned}
&\int_a^b dx \int_a^b dx'\, G^R_{xxxx}(t,x,0,0,\,t',x',0,m_iL) = \\
&\quad \frac{1}{8\pi^5 L^6}\prod_{i=1}^{6}\sum_{m_i=1}^{\infty}{}' \Bigg[-\frac{424\gamma^6}{3(\sum_{i=1}^6\ m_i^2)(\gamma^2+\sum_{i=1}^6\ m_i^2)^5} - \frac{68\gamma^8}{3(\sum_{i=1}^6\ m_i^2)^2(\gamma^2+\sum_{i=1}^6\ m_i^2)^5} \\
&\quad + \frac{390\gamma^4}{(\gamma^2+\sum_{i=1}^6\ m_i^2)^5} + \frac{195\gamma^2(\sum_{i=1}^6\ m_i^2)}{(\gamma^2+\sum_{i=1}^6\ m_i^2)^5} - \frac{4\gamma^{10}}{3(\sum_{i=1}^6\ m_i^2)^3(\gamma^2+\sum_{i=1}^6\ m_i^2)^5} \\
&\quad - \frac{400\gamma^5 + 100(\sum_{i=1}^6\ m_i^2)\gamma^3 + 15(\sum_{i=1}^4\ m_i^2)^2\gamma}{2(\sum_{i=1}^4\ m_i^2+\gamma^2)^{11/2}} \ln\left(\frac{\sqrt{\sum_{i=1}^6\ m_i^2+\gamma^2}+\gamma}{\sqrt{\sum_{i=1}^6\ m_i^2+\gamma^2}-\gamma}\right)\Bigg].
\end{aligned}
\tag{124}
$$

The summation is dominated by the second term when $\gamma \gg 1$ and thus is approximated by an integral as

$$
-\frac{1}{6\pi^5 L^6}\prod_{i=1}^{6}\int_{1/\gamma}^{\infty} dx_i\, \frac{1}{(\sum_{i=1}^6 x_i^2)^3(\sum_{i=1}^6 x_i^2+1)^5} \approx \frac{1}{6\pi^5 L^6}\ln(\gamma)\,. \tag{125}
$$

Hence, we have

$$
\langle \sigma_1^2\rangle_R \approx \frac{r^2}{48\pi^5 L^6}\ln(\gamma)\,, \tag{126}
$$

and

$$
\Delta t = \frac{\sqrt{\langle \sigma_1^2\rangle_R}}{r} \approx \frac{\sqrt{2}t_{pl}}{\sqrt{3}\pi^2}\sqrt{\ln(\gamma)}\,. \tag{127}
$$

f. The case with n=7 This is the 11 dimensional spacetime with 7 extra dimensions . The relevant two-point function can be shown to be

$$
\begin{aligned}
&G_{xxxx}(t,x,\mathbf{0},\,t',x',0,0,m_1L,...,m_7L) \\
&= \frac{1}{16\pi^5}\Big[\frac{72\rho^{12}}{\alpha_7^5(\rho^2+\alpha_7^2)^8} + \frac{1056\rho^{10}}{\alpha_7^3(\rho^2+\alpha_7^2)^8}\frac{14256\rho^8}{\alpha_7(\rho^2+\alpha_7^2)^8} \\
&\qquad - \frac{66528\alpha_7\rho^6}{(\rho^2+\alpha_7^2)^8} + \frac{27720\alpha_7^3\rho^4}{2(\rho^2+\alpha_7^3)^8}\Big].
\end{aligned}
\tag{128}
$$

Following the same steps, one finds for $\gamma \gg 1$

$$
\langle \sigma_1^2\rangle_R \approx \frac{5}{16\pi^5 L^7}\prod_{i=1}^{7}\int_{1/\gamma}^{\infty} dx_i\, \frac{1}{(\sum_{i=1}^7 x_i^2)^{\frac{7}{2}}(\sum_{i=1}^7 x_i^2+1)^6} \approx \frac{5}{16\pi^5 L^7}\ln(\gamma)\,, \tag{129}
$$

and

$$
\Delta t = \frac{\sqrt{\langle \sigma_1^2\rangle_R}}{r} \approx \frac{\sqrt{10}t_{pl}}{\pi^4}\sqrt{\ln(\gamma)}\,. \tag{130}
$$

Thus we find that in all of these models in which there is more than one flat compactified extra dimension, the lightcone fluctuation effect grows only logarithmically with distance. Hence we are unable to derive any constraints on these models. We have not been able to prove that this behavior holds for any number of flat extra dimensions greater than one, but conjecture that this is the case.

IV. PARALLEL BRANE-WORLDS SCENARIO

In this section, we examine the lightcone fluctuations due to the presence of two 3+1 dimensional hyperplane boundaries, i.e., " 3-branes", living in extra dimensions separated from each other by some distance. This framework is motivated by the recent proposal to resolve the unnatural hierarchy between the weak and Planck scales [5]. In this scenario, four-dimensional particle theory, such as the Standard Model, is confined to live in one of the branes, but gravity is free to propagate in the higher dimensional bulk. Therefore the bulk spacetime is dynamical, and the 3-branes can not be rigid, but must undergo quantum fluctuations in their positions, which we assume to be order of l_p, the Planck length in higher dimensions. To be more specific, we suppose one brane is located at the origin and the other at $(0, 0, 0, z_1, z_2, ..., z_n)$, and we shall examine the effect of lightcone fluctuations due to gravitons in the bulk by looking at a light ray traveling parallel to one of the boundaries, in the $x-$axis, for example, but separated from it by a distance $z \sim l_p$. This feature is to reflect the quantum fluctuations in the position of the brane [39]. We consider both the Neumann and Dirichlet boundary conditions for the graviton field at each brane hyperplane. Once we have the graviton two-point functions without any boundary, those with two parallel plane boundaries can also be found by the method of image sum. If gravitons satisfy the Dirichlet boundary condition, then the renormalized graviton two-point function is given by the following multiple image sum

$$
\begin{aligned}
&G^R_{\mu\nu\rho\sigma}(t, z_i,\, t', z'_i) \\
&\quad = \prod_{i=1}^{n} \sum_{m_i=-\infty}^{+\infty}{}' G_{\mu\nu\rho\sigma}(t, z_i,\, t', z'_i + 2m_iL) - \prod_{i=1}^{n} \sum_{m_i=-\infty}^{+\infty} G_{\mu\nu\rho\sigma}(t, z_i,\, t', -z'_i + 2m_iL) \\
&\quad = \prod_{i=1}^{n} \sum_{m_i=1}^{\infty} \Big[G_{\mu\nu\rho\sigma}(t, z_i,\, t', 2m_iL + z'_i) - G_{\mu\nu\rho\sigma}(t, z_i,\, t', 2m_iL - z'_i) \\
&\qquad\qquad + G_{\mu\nu\rho\sigma}(t, z_i,\, t', -2m_iL + z'_i) - G_{\mu\nu\rho\sigma}(t, z_i,\, t', -2m_iL - z'_i) \Big] \\
&\qquad\qquad - G_{\mu\nu\rho\sigma}(t, z_i,\, t', -z'_i)
\end{aligned} \tag{131}
$$

Again the prime denotes omitting the $m_i = 0$ term in the summation. For the Neumann boundary, the renormalized graviton two-point function becomes

$$
\begin{aligned}
&G^R_{\mu\nu\rho\sigma}(t, z_i,\, t', z'_i) \\
&\quad = \prod_{i=1}^{n} \sum_{m_i=-\infty}^{+\infty}{}' G_{\mu\nu\rho\sigma}(t, z_i,\, t', z'_i + 2m_iL) + \prod_{i=1}^{n} \sum_{m_i=-\infty}^{+\infty} G_{\mu\nu\rho\sigma}(t, z_i,\, t', -z'_i + 2m_iL)
\end{aligned}
$$

$$= \prod_{i=1}^{n} \sum_{m_i=1}^{\infty} \Big[G_{\mu\nu\rho\sigma}(t, z_i,\, t', 2m_iL + z_i') + G_{\mu\nu\rho\sigma}(t, z_i,\, t', 2m_iL - z_i')$$
$$+ G_{\mu\nu\rho\sigma}(t, z_i,\, t', -2m_iL + z_i') + G_{\mu\nu\rho\sigma}(t, z_i,\, t', -2m_iL - z_i') \Big]$$
$$+ G_{\mu\nu\rho\sigma}(t, z_i,\, t', -z_i') \,. \tag{132}$$

Let us examine a light ray propagating along the x-axis starting from $(a, 0, 0, z_1, ..., z_n)$ to $(b, 0, 0, z_1, ..., z_n)$. For simplicity, let us assume that $z_1 = \cdots = z_n = z \approx l_p/\sqrt{n}$. Then the mean squared fluctuation in the geodesic interval function is

$$\langle \sigma_1^2 \rangle = \frac{1}{8}(b-a)^2 \int_a^b dx \int_a^b dx' \; G^R_{xxxx}(t, x, 0, 0, z_i,\, t', x', 0, 0, z_i). \tag{133}$$

A. Five dimensional theory

Here we have one extra dimension, and the relevant two-point function is

$$G_{xxxx}(t, x, 0, 0, z_1, t', x', 0, 0, z_1') = \frac{1}{4\pi^2}\Bigg[-\frac{\rho^6}{(\rho^2 + \Delta z_1^2)^3 |\Delta z_1|^3} - \frac{7\rho^4}{(\rho^2 + \Delta z_1^2)^3 |\Delta z_1|}$$
$$+ \frac{9\rho^2 |\Delta z_1|}{(\rho^2 + \Delta z_1^2)^3} - \frac{|\Delta z_1|^3}{(\rho^2 + \Delta z_1^2)^3} \Bigg] , \tag{134}$$

where $\Delta z_1 = z_1 - z_1'$. We then have

$$\int_a^b dx \int_a^b dx'\, G_{xxxx}(t, x, 0, 0, z_1, t', x', 0, 0, z_1')$$
$$= \frac{1}{\pi^2 |\Delta z_1|} \ln(1 + \frac{r^2}{\Delta z_1^2}) - \frac{r^2}{4\pi^2 |\Delta z_1|^3} - \frac{r^2}{\pi^2 (r^2 + \Delta z_1^2)|\Delta z_1|}$$
$$\equiv f(\Delta z_1, r) \,. \tag{135}$$

If gravitons satisfy the Dirichlet boundary conditions, it follows that

$$\langle \sigma_1^2 \rangle = \frac{1}{8} r^2 \int_a^b dx \int_a^b dx' \; G^R_{xxxx}(t, x, 0, 0, z_i,\, t', x', 0, z_i)$$
$$= -\frac{1}{8} r^2 f(2z, r) + \frac{1}{8} r^2 \sum_{m=1}^{\infty} \left[2f(2mL, r) - f(2z - 2mL, r) - f(2z + 2mL, r) \right] . \tag{136}$$

Here we are interested in the case in which $r \gg z$ and $L \gg z$. One then finds that

$$f(2z, r) \approx -\frac{1}{L} \frac{r^2}{32\pi^2 z^2} \frac{L}{z} = -\frac{1}{L} \frac{\gamma^2}{32\pi^2} \left(\frac{L}{z} \right)^3 , \tag{137}$$

and

$$
\begin{aligned}
\sum_{m=1}^{\infty} & \left[2f(2mL,r) - f(2z-2mL,r) - f(2z+2mL,r)\right] \\
&\approx \frac{1}{32\pi^2}\sum_{m=1}^{\infty}\left[-\frac{2r^2}{m^3L^3} + \frac{r^2}{(mL+z)^3} + \frac{r^2}{(mL-z)^3}\right] \\
&= \frac{1}{32\pi^2 L}\sum_{m=1}^{\infty}\left[-\frac{2\gamma^2}{m^3} + \frac{\gamma^2}{(m-z/L)^3} + \frac{\gamma^2}{(m+z/L)^3}\right] \\
&\approx \frac{\gamma^2}{32\pi^2 L}\sum_{m=1}^{\infty}\left[\frac{12}{m^5}\left(\frac{z}{L}\right)^2 + \frac{30}{m^7}\left(\frac{z}{L}\right)^4 + O((z/L)^6)\right] \\
&\approx \frac{\gamma^2}{32\pi^2 L}\sum_{m=1}^{\infty}\frac{12}{m^5}\left(\frac{z}{L}\right)^2 = \frac{3\gamma^2\zeta(5)}{8\pi^2 L}\left(\frac{z}{L}\right)^2 .
\end{aligned}
\tag{138}
$$

Here $\zeta(5)$ is the Riemann-zeta function. Consequently, we find

$$
\langle\sigma_1^2\rangle = \frac{\gamma^2 r^2}{8^2\pi^2 L}\left[\frac{1}{4}\left(\frac{L}{z}\right)^3 + 3\zeta(5)\left(\frac{z}{L}\right)^2\right], \tag{139}
$$

thus,

$$
\Delta t = \frac{\sqrt{\langle\sigma_1^2\rangle_R}}{r} \approx \gamma\, t_{pl}\sqrt{\frac{1}{2\pi}\left[\frac{1}{4}\left(\frac{L}{z}\right)^3 + 3\zeta(5)\left(\frac{z}{L}\right)^2\right]} \approx \frac{1}{2\sqrt{2\pi}}\left(\frac{L}{z}\right)^{\frac{3}{2}}\left(\frac{r}{L}\right) t_{pl} . \tag{140}
$$

If instead of the Dirichlet boundary condition, gravitons are forced to satisfy the Neumann boundary condition, one finds

$$
\langle\sigma_1^2\rangle = -\frac{\gamma^2 r^2}{8^2\pi^2 L}\left[\frac{1}{4}\left(\frac{L}{z}\right)^3 + \zeta(3) + 3\zeta(5)\left(\frac{z}{L}\right)^2\right], \tag{141}
$$

and

$$
\Delta t = \frac{\sqrt{|\langle\sigma_1^2\rangle_R|}}{r} \approx \gamma\, t_{pl}\sqrt{\frac{1}{2\pi}\left[\frac{1}{4}\left(\frac{L}{z}\right)^3 + \zeta(3) + 3\zeta(5)\left(\frac{z}{L}\right)^2\right]} \approx \frac{1}{2\sqrt{2\pi}}\left(\frac{L}{z}\right)^{\frac{3}{2}}\left(\frac{r}{L}\right) t_{pl} . \tag{142}
$$

Here we find that the choice of different boundary conditions has very little effect on the growth of the lightcone fluctuations as long as $L \gg z$, although the fluctuations are slightly larger in the case of the Neumann boundary condition. Comparing this Δt with that in the case of periodical compactification Eq. (50), one can see that the effect of light cone fluctuation here is larger for a given size of extra dimensions, L. Another very distinctive feature is that here Δt increases as L increases provided that L and z are independent of each other, which is in sharp contrast with the case of periodical compactification. As before, the fluctuation in the flight time of pulses, Δt, can be applied to the successive wave crests of a plane wave. This leads to a broadening of spectral lines from luminous sources. In case of periodical compactification, we used the data from gamma ray bursters to derive a very strong lower bound on L. In the present framework, however, we have an intrinsic lower bound $L \geq z$. Recall that z is the Planck length in higher dimensions which may be much larger than that in 4 dimensions. In the picture of Ref. [5], z and L are not independent, and may be expressed in terms of the Planck mass in $4+n$ dimensions, $M_p^{(4+n)}$, as

$$l_p = 2 \times 10^{-17} \Big(\frac{1Tev}{M_p^{(4+n)}} \Big) \, \mathrm{cm} \, , \tag{143}$$

$$L = 10^{30/n-17} \left(\frac{1Tev}{M_p^{(4+n)}} \right)^{1+2/n} \mathrm{cm} \, . \tag{144}$$

One has for the ratio L/z

$$\frac{L}{z} = \frac{L}{l_p} = 0.5 \times 10^{30/n} \Big(\frac{1Tev}{M_p^{(4+n)}} \Big)^{2/n} \, . \tag{145}$$

Express Eq. (140) or Eq. (142) as

$$\Delta t \approx \frac{1}{2\sqrt{2\pi}} \Big(\frac{L}{z} \Big)^{\frac{1}{2}} \Big(\frac{r}{z} \Big) \, t_{pl} \, , \tag{146}$$

or equivalently,

$$\Delta t \approx 10^{15} \Big(\frac{1Tev}{M_p^{(4+n)}} \Big)^{1/n-1} \frac{r}{10^{-17} cm} \, t_{pl} = 10^{32} \left(\frac{1Tev}{M_p^{(4+n)}} \right)^{1/n-1} \left(\frac{r}{1 \, cm} \right) t_{pl} \, . \tag{147}$$

Let $M_p^{(4+n)}$ be $\sim$ 1Tev. Then for an astronomical source of cosmological distance, such as gamma-ray bursters with redshift ~ 1, or $r \sim 10^{28}$cm, we get the following estimate for the $n = 1$ model

$$\Delta t \sim 10^{60} \times 10^{-44} s = 10^{16} s. \tag{148}$$

That is far too large, so that the $n = 1$ model can be ruled out completely.

B. Higher dimensional theories

We shall only study the 6 dimensional case in some detail, but shall list results for other cases. It is not difficult to see that the relevant two-point function can be obtained from Eq. (96) by replacing α_2 there with $\sum_{i=1}^{2} \Delta z_i^2$. Therefore one easily finds from Eq. (101)

$$\begin{aligned} &\int_a^b dx \int_a^b dx' \, G_{xxxx}(t, x, 0, 0, z_1, z_2, \, t', x', 0, 0, z_1', z_2') \equiv f_2(\Delta z_1, \Delta z_2, r) \\ &\quad = \frac{1}{4\pi^3} \Bigg[\frac{r^2(\Delta z_1^2 + \Delta z_2^2)}{(r^2 + \Delta z_1^2 + \Delta z_2^2)^3} + \frac{10 r^4}{3(r^2 + \Delta z_1^2 + \Delta z_2^2)^3} \\ &\qquad - \frac{8 r^6}{3(\Delta z_1^2 + \Delta z_2^2)(r^2 + \Delta z_1^2 + \Delta z_2^2)^3} \\ &\qquad - \frac{8 r^5 + 4(\Delta z_1^2 + \Delta z_2^2) r^3 + (\Delta z_1^2 + \Delta z_2^2)^2 r}{2(\Delta z_1^2 + \Delta z_2^2 + r^2)^{7/2}} \ln \left(\frac{\sqrt{\Delta z_1^2 + \Delta z_2^2 + r^2} + r}{\sqrt{\Delta z_1^2 + \Delta z_2^2 + r^2} - r} \right) \Bigg] . \end{aligned} \tag{149}$$

Then it follows that

$$
\begin{aligned}
\langle\sigma_1^2\rangle &= \frac{1}{8}r^2\int_a^b dx\int_a^b dx'\ G^R_{xxxx}(t,x,0,0,z,z,\,t',x',0,z,z) \\
&= -\frac{1}{8}r^2 f(2z,2z,r)+\frac{1}{8}r^2\sum_{m_1=1}^{\infty}\sum_{m_2=1}^{\infty}[2f(2m_1L,2m_2L,r) \\
&\quad -f(2z-2m_1L,2z-m_2L,r)-f(2z+2m_1L,2z+2m_2L,r)]\,.
\end{aligned}
\tag{150}
$$

Here we are interested in the case when $r\gg z$ and $L\gg z$. One finds that

$$
f(2z,2z,r)\approx -\frac{1}{6\pi^3L^2}\left(\frac{L}{z}\right)^2-\frac{4}{\pi^3L^2}\frac{\ln(r/z)}{\gamma^2}\,,
\tag{151}
$$

and

$$
\begin{aligned}
g(r,z,L) &\equiv \sum_{m_1=1}^{\infty}\sum_{m_2=1}^{\infty}[\,2f(2m_1L,2m_2L,r)-f(2z-2m_1L,2z-2m_2L,r) \\
&\quad -f(2z+2m_1L,2m_2L,r)\,] \\
&\approx \frac{1}{4\pi^3L^2}\sum_{m_1=1}^{\infty}\sum_{m_2=1}^{\infty}\left(-\frac{4\gamma^6}{3(m_1^2+m_2^2)(\gamma+m_1^2+m_2^2)^3}\right. \\
&\quad +\frac{2\gamma^6}{3[(m_1+z/L)^2+(m_2+z/L)^2][\gamma^2+(m_1+z/L)^2+(m_2+z/L)^2]^3} \\
&\quad \left.+\frac{2\gamma^6}{3[(m_1-z/L)^2+(m_2-z/L)^2][\gamma^2+(m_1-z/L)^2+(m_2-z/L)^2]^3}\right).
\end{aligned}
\tag{152}
$$

If we assume the travel distance r is much larger than the size of the extra dimensions L, i.e $\gamma\gg 1$, $g(r,z,L)$ can be approximated by integrals as follows

$$
\begin{aligned}
g(r,z,L) &\approx -\frac{1}{6\pi^3L^2}\int_{1/\gamma}^{\infty}dx_1\int_{1/\gamma}^{\infty}dx_2\,\frac{2}{(x_1^2+x_2^2)(x_1^2+x_2^2+1)^3} \\
&\quad +\frac{1}{6\pi^3L^2}\int_{\frac{1}{\gamma}(1-z/L)}^{\infty}dx_1\int_{\frac{1}{\gamma}(1-z/L)}^{\infty}dx_2\,\frac{1}{(x_1^2+x_2^2)(x_1^2+x_2^2+1)^3} \\
&\quad +\frac{1}{6\pi^3L^2}\int_{\frac{1}{\gamma}(1+z/L)}^{\infty}dx_1\int_{\frac{1}{\gamma}(1+z/L)}^{\infty}dx_2\,\frac{1}{(x_1^2+x_2^2)(x_1^2+x_2^2+1)^3} \\
&\approx \frac{1}{6\pi^3L^2}\left[2\ln(\gamma)+\ln\left(\frac{1-z/L}{\gamma}\right)+\ln\left(\frac{1+z/L}{\gamma}\right)\right] \\
&= \frac{1}{6\pi^3L^2}\ln\Big[1-(z/L)^2\Big]\,.
\end{aligned}
\tag{153}
$$

Therefore,

$$
\begin{aligned}
\langle\sigma_1^2\rangle &= \frac{1}{8}r^2\int_a^b dx\int_a^b dx'\ G^R_{xxxx}(t,x,0,0,z,z,\,t',x',0,z,z)\approx -\frac{r^2}{8}f(2z,2z,r) \\
&\approx \frac{r^2}{48\pi^3L^2}\left(\frac{L}{z}\right)^2+\frac{2r^2}{3\pi^3L^2}\frac{\ln(r/z)}{\gamma^2}\,.
\end{aligned}
\tag{154}
$$

Hence,

$$\Delta t = \frac{\sqrt{\langle \sigma_1^2 \rangle_R}}{r} \approx \sqrt{\frac{2}{3\pi^2}} \left(\frac{L}{z} \right) . \tag{155}$$

Notice here that the leading term does not depend on the travel distance r and moreover the next order corrections decrease as r increases. Recall that $\gamma = r/L$. Interestingly, the lightcone fluctuation grows as the compactification scale increases.

However, if we change to the Neumann boundary condition, the behavior will be different as we can see from the following analysis. Let us note that in this case

$$\begin{aligned} \langle \sigma_1^2 \rangle &= \frac{1}{8} r^2 \int_a^b dx \int_a^b dx' \; G^R_{xxxx}(t, x, 0, 0, z, z, \; t', x', 0, z, z) \\ &= \frac{1}{8} r^2 f(2z, 2z, r) + \frac{1}{8} r^2 \sum_{m_1=1}^{\infty} \sum_{m_2=1}^{\infty} \left[2 f(2m_1 L, 2m_2 L, r) \right. \\ &\quad \left. + f(2z - 2m_1 L, 2z - m_2 L, r) + f(2z + 2m_1 L, 2z + 2m_2 L, r) \right] . \end{aligned} \tag{156}$$

Notice the sign changes in the above expression as compared to Eq (150). Then it follows that

$$\begin{aligned} \langle \sigma_1^2 \rangle &\approx -\frac{r^2}{48\pi^3 L^2} \left(\frac{L}{z} \right)^2 - \frac{2r^2}{3\pi^3 L^2} \frac{\ln(r/z)}{\gamma^2} + \frac{r^2}{48\pi^3 L^2} \left[2 \ln(\gamma) - \ln\left(\frac{1 - z^2/L^2}{\gamma^2} \right) \right] \\ &\approx \frac{r^2}{12\pi^3 L^2} \ln(\gamma) - \frac{r^2}{48\pi^3 L^2} \left(\frac{L}{z} \right)^2 , \end{aligned} \tag{157}$$

and

$$\Delta t = \frac{\sqrt{\langle \sigma_1^2 \rangle_R}}{r} \approx \sqrt{\left| \frac{2}{3\pi^2} \left(\frac{L}{z} \right)^2 - \frac{8}{3\pi^2} \ln(r/L) \right|} \; t_{pl} \, . \tag{158}$$

Here we have two different contributing terms to the mean deviation in time; one is independent of r but increases as L increases, the other grows logarithmically as r increase or L decreases.

Similarly, we have calculated cases up to $n = 7$ as motivated by string/M theory. The results are (all in units of t_{pl}, the Planck time in four dimensions):

For n=3,

$$\begin{aligned} \Delta t &\approx \sqrt{\frac{1}{6\pi^3}} \left(\frac{L}{z} \right)^{\frac{3}{2}} , && \text{for Dirichlet boundary condition} \\ \Delta t &\approx \sqrt{\left| \frac{1}{6\pi^3} \left(\frac{L}{z} \right)^3 - \frac{2}{3\pi^3} \ln(r/L) \right|} , && \text{for Neumann boundary condition.} \end{aligned} \tag{159}$$

For n=4,

$$\Delta t \approx \sqrt{\frac{5}{12\pi^4}}\left(\frac{L}{z}\right)^2, \qquad \text{for Dirichlet boundary condition}$$

$$\Delta t \approx \sqrt{\left|\frac{5}{12\pi^4}\left(\frac{L}{z}\right)^4 - \frac{5}{3\pi^4}\ln(r/L)\right|}, \qquad \text{for Neumann boundary condition.} \tag{160}$$

For n=5,

$$\Delta t \approx \frac{1}{4\pi^2}\left(\frac{L}{z}\right)^{\frac{5}{2}}, \qquad \text{for Dirichlet boundary condition}$$

$$\Delta t \approx \sqrt{\left|\frac{1}{16\pi^4}\left(\frac{L}{z}\right)^5 - \frac{1}{4\pi^4}\ln(r/L)\right|}, \qquad \text{for Neumann boundary condition.} \tag{161}$$

For n=6,

$$\Delta t \approx \frac{1}{8\sqrt{3}\pi^{\frac{5}{2}}}\left(\frac{L}{z}\right)^3, \qquad \text{for Dirichlet boundary condition}$$

$$\Delta t \approx \frac{1}{8}\sqrt{\left|\frac{1}{3\pi^5}\left(\frac{L}{z}\right)^6 - \frac{4}{3\pi^5}\ln(r/L)\right|}, \qquad \text{for Neumann boundary condition.} \tag{162}$$

For n=7,

$$\Delta t \approx \frac{1}{8}\sqrt{\frac{5}{2\pi^5}}\left(\frac{L}{z}\right)^{\frac{7}{2}}, \qquad \text{for Dirichlet boundary condition}$$

$$\Delta t \approx \frac{1}{8}\sqrt{\left|\frac{5}{2\pi^5}\left(\frac{L}{z}\right)^7 - \frac{10}{\pi^5}\ln(r/L)\right|}, \qquad \text{for Neumann boundary condition.} \tag{163}$$

A few comments are now in order for all the cases we have examined: First, to the leading order Δt does not grow as r increases for the Dirichlet boundary condition. Second, Δt behaves differently for different boundary conditions, but the logarithmic dependence associated with the Neumann boundary condition can be neglected as far as observation is concerned because logarithmic growth is extremely small. Third, our results seem to suggest a leading order behavior of $(L/z)^{n/2}$ for any dimensions. Finally, our results are quite different from those obtained by Ref. [40] where only the contribution of the F term was considered and only one dominant term in the sums computed. Our results are much smaller than those obtained by these authors because of cancellations among the various terms.

We wish to gain an understanding of how large the effect of lightcone fluctuations can be in these higher dimensions and whether they might be observable.

From Eq. (145) and the above results, one finds

$$\Delta t \sim 10^{15} t_{pl}\left(\frac{1Tev}{M_p^{(4+n)}}\right). \tag{164}$$

This result reveals that the smaller is $M_p^{(4+n)}$, the larger is the effect of lightcone fluctuations. According to Ref. [5], $M_p^{(4+n)}$ may as low as the order of one Tev. This leads to

$$\Delta t \sim 10^{-29} s\,. \tag{165}$$

This is a tiny effect by conventional standards. However if we note that the above mean time deviation is equivalent to an uncertainty in position

$$\Delta x \sim 10^{-21}\, m \tag{166}$$

and the operation of gravity-wave interferometers is based upon the detection of minute changes in the positions of some test masses (relative to the position of a beam splitter), we can see that this effect might be testable in the next LIGO/VIRGO generation of gravity-wave interferometers [41,42]. It constitutes an additional source of noise due to quantum gravity. Currently, the sensitivity of these gravity-wave interferometers has already reached the order of 10^{-19}m [43]. Note that Amelino-Camelia [44] and Ng and van Dam [45] have also proposed rather different quantum gravity effects which might also be detectable by laser interferometers.

A more complete discussion of the observability of the Δt given by Eq. (164) should involve a calculation of the correlation time, analogous to that performed in Sect. III A 3 for the five dimensional Kaluza-Klein model. This calculation has not yet been performed. However, it is reasonable to guess that the result will be of the order of or less than Tev scale which characterizes this model. Recall that in the five dimensional compactified model, the correlation time was found to be much smaller than the compactification scale when the travel distance is large. If this guess is correct, then the correlation time is much smaller than Δt itself.

V. DISCUSSION AND CONCLUSIONS

In this paper, we have examined the effects of compactified extra dimensions upon the propagation of light in the uncompactified dimensions. There are nontrivial effects that arise from quantum fluctuations of the gravitational field, induced by the compactification. These effects take the form of lightcone fluctuations, variations in the flight times of pulses between a source and a detector. The crucial quantity describing these fluctuations is Δt, the expected variation in arrival times of two successive pulses which are separated by more than a correlation time. In Sect. II B, we gave a derivation of the formula for Δt using the geodesic deviation equation. This derivation allowed us to discuss issues of gauge and Lorentz invariance. In particular, it demonstrates that Δt is gauge invariant. All of the subsequent explicit calculations are performed in the transverse-tracefree gauge.

As a prelude, we found the graviton two-point function in this gauge in Minkowski spacetime of arbitrary dimension. The two-point function in a higher dimensional flat compactified spacetime is given as an image sum. We then calculated Δt in the five-dimensional Kaluza-Klein model, five-dimensional flat spacetime with one periodic spatial dimension. We found that Δt grows linearly with increasing distance between the source and the detector, and is inversely proportional to the compactification length, L. This result differs from the square root dependence on distance that was found in four-dimensional flat spacetime with one periodic spatial dimension [11]. This demonstrates that lightcone fluctuation effects are rather model-dependent. We also calculated the correlation time in the five-dimensional model and found that it is typically small compared to the compactification scale L. This allows us to place very tight constraints on the parameters of this model.

We favor the viewpoint that the lightcone fluctuation effects should vanish only in the limit that $L \to \infty$. If one adopts this view, then the five-dimensional Kaluza-Klein model can be ruled out. Data from gamma ray burst sources imply a lower bound on L which is larger than upper bounds obtained from other considerations. However, another logical possibility is that lightcone fluctuation effects happen to vanish ($\langle \sigma_1^2 \rangle = 0$) at the present compactification scale. Even if one adopts this viewpoint, one still obtains very strong constraints on the fractional change in L which can have occurred over a cosmological time scale. This in turn tightly constrains any five-dimensional Kaluza-Klein cosmology.

We also examined an alternative five-dimensional model (the brane worlds scenario) in which gravitons satisfy Dirichlet or Neumann boundary conditions on a pair of parallel four-dimensional branes (one of which represents our world). In this model, we again find that the lightcone fluctuations are so large that the model can be ruled out.

We next turned our attention to models with more than one extra dimension. In the case of two or more flat, periodically compactified dimensions, we found that Δt grows only logarithmically with distance, and that no constraints may be placed on these models. In the case of the brane worlds scenario with more than one extra dimension, we found that Δt approaches a constant which can be of the order of $10^{-29}s$. This would produce a source of noise in laser interferometer detectors of gravity waves, which may eventually be within their range of sensitivity.

In summary, compactified extra dimensions have the possibility to produce observable effects by enhancing the quantum fluctuations of the gravitational field. These effects might be used either to place constraints on theories with extra dimensions, or else possibly eventually to provide positive evidence for the existence of extra dimensions. Although our results for more than one flat compactified extra dimension are too weak for either of these purposes, the required calculations for models with curved extra dimensions have not yet been performed. Another model which has not yet been examined in this context is the Randall-Sundrum model [46], with an uncompactified fifth dimension. In this latter model, propagating graviton modes are effectively confined within a finite volume in the fifth dimension, so one might expect nonzero lightcone fluctuations.

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APPENDIX

VI. GRAVITON TWO-POINT FUNCTIONS IN SPACETIME WITH ARBITRARY NUMBER OF EXTRA DIMENSIONS

Here we evaluate the functions $D^n(x.x')$, $F^n_{ij}(x,x')$ and $H^n_{ijkl}(x,x')$ defined in Eqs. (27), (26) and (28), respectively. Once these functions are given, the graviton two point functions are easy to obtain. Define

$$R = |\mathbf{x} - \mathbf{x}'|, \quad \Delta t = t - t', \quad k = |\mathbf{k}| = \omega\,, \tag{A1}$$

and assume n extra dimensions, then

$$\begin{aligned} D^n(x,x') &= \frac{Re}{(2\pi)^{3+n}} \int \frac{d^{(3+n)}\mathbf{k}}{2\omega} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} e^{-i\omega(t-t')} \\ &= \frac{Re}{2(2\pi)^{3+n}} \int_0^\infty k^{n+1} e^{-ik\Delta t}\, dk \int_0^\pi d\theta_1\, \sin^{1+n}\theta_1 e^{ikR\cos\theta_1} \\ &\quad \times \int_0^\pi d\theta_2\, \sin^n\theta_2 ... \int_0^\pi d\theta_{n+1}\, \sin\theta_{n+1} \int_0^{2\pi} d\theta_{n+2} \\ &= \frac{a_n\, Re}{2(2\pi)^{3+n}} \int_0^\infty k^{n+1} e^{-ik\Delta t}\, dk \int_{-1}^1 e^{ikRx} (1-x^2)^{n/2}\, dx \\ &= \frac{a_n\, Re}{(2\pi)^{3+n}} \int_0^\infty k^{n+1} e^{-ik\Delta t}\, dk \int_0^1 (1-x^2)^{n/2} \cos(kRx)\, dx \\ &= \frac{a_n \sqrt{\pi} 2^{\frac{n+1}{2}} \Gamma(\frac{n}{2}+1)\, Re}{2(2\pi)^{3+n}} \frac{1}{R^{\frac{n+1}{2}}} \int_0^\infty k^{\frac{n+1}{2}} J_{\frac{n+1}{2}}(kR) e^{-ik\Delta t}\, dk \\ &= \frac{a_n \sqrt{\pi} 2^{\frac{n+1}{2}} \Gamma(\frac{n}{2}+1)\, Re}{2(2\pi)^{3+n}} \frac{1}{R^{\frac{n+1}{2}}} \lim_{\alpha\to 0^+ + i\Delta t} \int_0^\infty k^{\frac{n+1}{2}} J_{\frac{n+1}{2}}(kR) e^{-\alpha k}\, dk \\ &= \frac{a_n 2^n \Gamma(\frac{n}{2}+1)^2}{(2\pi)^{3+n}} \frac{1}{(R^2-\Delta t^2)^{n/2+1}} = \frac{\Gamma(\frac{n}{2}+1)}{4\pi^{\frac{n+4}{2}}} \frac{1}{(R^2-\Delta t^2)^{n/2+1}}\,. \end{aligned} \tag{A2}$$

Here we have defined

$$a_n = \int_0^\pi d\theta_2\, \sin^n\theta_2 ... \int_0^\pi d\theta_{n+1}\, \sin\theta_{n+1} \int_0^{2\pi} d\theta_{n+2} = \frac{2\pi^{\frac{n}{2}+1}}{\Gamma(\frac{n}{2}+1)}\,, \tag{A3}$$

and used

$$\int_0^1 \cos(kRx)(1-x^2)^{n/2}\, dx = \frac{\sqrt{\pi}}{2} \Gamma(\frac{n}{2}+1) \Big(\frac{2}{kR}\Big)^{\frac{n+1}{2}} J_{\frac{n+1}{2}}(kR)\,, \tag{A4}$$

and

$$\int_0^\infty e^{-\alpha x} J_\nu(\beta x) x^\nu\, dx = \frac{(2\beta)^\nu \Gamma(\nu+1/2)}{\sqrt{\pi}(\alpha^2+\beta^2)^{\nu+\frac{1}{2}}}\,, \qquad Re\nu > -1/2\,. \tag{A5}$$

When n is odd, $D^n(x,x')$ should be taken to be zero when $R^2 < \Delta t^2$.

Let us now turn our attention to the calculation of F_{ij} and H_{jikl}. We find

$$
\begin{aligned}
F^n_{ij}(x,x') &= \frac{Re}{(2\pi)^{3+n}}\int d^{3+n}\mathbf{k}\frac{k_ik_j}{2\omega^3}e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}e^{-i\omega(t-t')} \\
&= \frac{Re}{2(2\pi)^{3+n}}\partial_i\partial'_j\int_0^\infty k^{n-1}e^{-ik\Delta t}\,dk\int_0^\pi d\theta_1\,\sin^{1+n}\theta_1 e^{ikR\cos\theta_1} \\
&\quad\times\int_0^\pi d\theta_2\,\sin^n\theta_2...\int_0^\pi d\theta_{n+1}\,\sin\theta_{n+1}\int_0^{2\pi} d\theta_{n+2} \\
&= \frac{a_n\,Re}{(2\pi)^{3+n}}\,\partial_i\partial'_j\int_0^\infty k^{n-1}e^{-ik\Delta t}\,dk\int_0^1 (1-x^2)^{n/2}\cos(kRx)\,dx \\
&= \frac{a_n\sqrt{\pi}2^{\frac{n+1}{2}}\Gamma(\frac{n}{2}+1)\,Re}{2(2\pi)^{3+n}}\,\partial_i\partial'_j\left(\frac{1}{R^{\frac{n+1}{2}}}\int_0^\infty k^{\frac{n-3}{2}}J_{\frac{n+1}{2}}(kR)e^{-ik\Delta t}\,dk\right) \\
&= \frac{a_n\sqrt{\pi}2^{\frac{n+1}{2}}\Gamma(\frac{n}{2}+1)}{2(2\pi)^{3+n}}\,\partial_i\partial'_j\left(\frac{Re}{R^{\frac{n+1}{2}}}\int_0^\infty k^{\frac{n-3}{2}}J_{\frac{n+1}{2}}(kR)e^{-ik\Delta t}\,dk\right) \\
&= \frac{Re}{2(2\pi)^{\frac{3+n}{2}}}\,\partial_i\partial'_j\left(\frac{1}{R^{\frac{n+1}{2}}}\int_0^\infty k^{\frac{n-3}{2}}J_{\frac{n+1}{2}}(kR)e^{-ik\Delta t}\,dk\right) \\
&= \frac{Re}{2(2\pi)^{\frac{3+n}{2}}}\,\partial_i\partial'_j\Bigg(\frac{n-1}{R^2}\frac{1}{R^{\frac{n-1}{2}}}\int_0^\infty k^{\frac{n-5}{2}}J_{\frac{n-1}{2}}(kR)e^{-ik\Delta t}\,dk \\
&\quad-\frac{1}{R^{\frac{n+1}{2}}}\int_0^\infty k^{\frac{n-3}{2}}J_{\frac{n-3}{2}}(kR)e^{-ik\Delta t}\,dk\Bigg)\,,
\end{aligned}
\tag{A6}
$$

where we have utilized a recursive formula for Bessel functions

$$
zJ_{\nu-1}(z)+zJ_{\nu+1}(z)=2\nu J_\nu(z)\,. \tag{A7}
$$

Similarly, one finds that

$$
\begin{aligned}
H^n_{ijkl}(x,x') &= \frac{Re}{(2\pi)^{3+n}}\int d^{3+n}\mathbf{k}\frac{k_ik_jk_kk_l}{2\omega^5}e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}e^{-i\omega(t-t')} \\
&= \frac{Re}{2(2\pi)^{\frac{3+n}{2}}}\,\partial_i\partial'_j\partial_k\partial_l\Bigg(\frac{n-1}{R^2}\frac{1}{R^{\frac{n-1}{2}}}\int_0^\infty k^{\frac{n-9}{2}}J_{\frac{n-1}{2}}(kR)e^{-ik\Delta t}\,dk \\
&\quad-\frac{1}{R^2}\frac{1}{R^{\frac{n+1}{2}}}\int_0^\infty k^{\frac{n-7}{2}}J_{\frac{n-3}{2}}(kR)e^{-ik\Delta t}\,dk\Bigg)\,.
\end{aligned}
\tag{A8}
$$

To proceed further with the calculation, we need to deal with the cases when n is odd or even separately.

A. The case of odd n

Assume $n=2m+1$ and define

$$
S(m)=\frac{Re}{R^{m+1}}\int_0^\infty k^{m-1}J_{m+1}(kR)e^{-ik\Delta t}\,dk\,,\quad m\geq 0\,, \tag{A9}
$$

$$
\begin{aligned}
T(m-1) &= \frac{Re}{R^{m+1}} \int_0^\infty k^{m-1} J_{m-1}(kR) e^{-ik\Delta t}\, dk \\
&= \frac{Re}{R^{m+1}} \lim_{\alpha\to 0^+ + i\Delta t} \int_0^\infty k^{m-1} J_{m-1}(kR) e^{-alphak}\, dk \\
&= \frac{2^{m-1}\Gamma(m-1/2)}{\sqrt{\pi}} \frac{\sqrt{R^2-\Delta t^2}}{R^2(R^2-\Delta t^2)^m}\,, \\
&= \frac{(2m-1)!!}{(2m-1)} \frac{\sqrt{R^2-\Delta t^2}}{R^2(R^2-\Delta t^2)^m}\,, \quad m \geq 1\,,
\end{aligned}
\tag{A10}
$$

where we have appealed to integral (6.623.1) in Ref. [47].The above result holds for $R^2 > \Delta t^2$, and $T(m-1)$ is zero when $R^2 < \Delta t^2$. Then it follows from Eq (A6) that

$$
F_{ij}^{2m+1} = \frac{1}{2(2\pi)^{m+2}} \partial_i \partial'_j \Big(S(m) \Big)\,,
\tag{A11}
$$

and

$$
S(m) = \frac{2m}{R^2} S(m-1) - T(m-1)
\tag{A12}
$$

Using the recursive relation Eq (A12), we can show that

$$
\begin{aligned}
S(m) &= \frac{(2m)!!}{R^{2m}} S(0) - \sum_{k=1}^{m} \frac{(2m)!!}{(2k)!!} \frac{T(k-1)}{R^{2m-2k}} \\
&= \frac{(2m)!!}{R^{2m}} S(0) \left[1 - \sum_{k=1}^{m} \frac{(2k-1)!!}{(2k)!!(2k-1)} \frac{R^{2k}}{(R^2-\Delta t^2)^k} \right] \\
&= -\frac{(2m)!!}{R^{2m}} S(0) \sum_{k=0}^{m} \frac{(2k+1)!!}{(2k)!!(2k+1)(2k-1)} \frac{R^{2k}}{(R^2-\Delta t^2)^k}\,.
\end{aligned}
\tag{A13}
$$

Here

$$
S(0) = \begin{cases} \frac{\sqrt{R^2-\Delta t^2}}{R^2} & \text{for } R^2 > \Delta t^2\,, \\ 0 & \text{for } R^2 < \Delta t^2\,, \end{cases}
\tag{A14}
$$

as reported in Ref. [12]. For the sake of completeness, we give its derivation below

$$
\begin{aligned}
S(0) &= \frac{Re}{R} \lim_{\alpha\to 0^+ + i\Delta t} \int_0^\infty \frac{J_1(kR)}{k} e^{-\alpha k} dk = \frac{1}{R} \int_0^\infty \frac{J_1(kR)\cos(k\Delta t)}{k}\, dk \\
&= \begin{cases} \frac{1}{R}\cos\Big(\arcsin(\Delta t/R)\Big) = \frac{\sqrt{R^2-\Delta t^2}}{R^2} & \text{for } R^2 > \Delta t^2\,, \\ 0 & \text{for } R^2 < \Delta t^2\,. \end{cases}
\end{aligned}
\tag{A15}
$$

If we define

$$
Q(m) = \frac{Re}{R^{m+1}} \int_0^\infty k^{m-3} J_{m+1}(kR) e^{-ik\Delta t}\, dk\,, \quad m \geq 0\,,
\tag{A16}
$$

then it is easy to see that

$$H^{2m+1}_{ijkl} = \frac{1}{2(2\pi)^{m+2}}\,\partial_i\partial'_j\partial_k\partial'_l\Big(Q(m)\Big)\,, \tag{A17}$$

and

$$Q(m) = \frac{2m}{R^2}Q(m-1) - \frac{1}{R^2}S(m-2)\,. \tag{A18}$$

The above equation applies for $m \geq 2$. To use it to get a general expression, we need $Q(0)$, which is given by

$$\begin{aligned} Q(0) &= \frac{1}{R}\int_0^\infty \frac{1}{k^3}J_1(kR)\cos(k\Delta t)\,dk \\ &= \lim_{\beta\to 0}\frac{1}{R}\int_0^\infty \frac{k^{-1}}{(k^2+\beta^2)}J_1(kR)\cos(k\Delta t)\,dk \\ &= \lim_{\beta\to 0}\frac{1}{R}\frac{e^{-\beta\Delta t}I_1(\beta R)}{\beta^2} = \frac{1}{2\beta} - \frac{1}{2\Delta t} \end{aligned} \tag{A19}$$

This leads to a vanishing H_{ijkl}. We next need $Q(1)$, which can be calculated, using integral (6.693.5) in Ref. [47], as follows

$$\begin{aligned} Q(1) &= \frac{1}{R^2}\int_0^\infty \frac{J_2(Rk)\cos(\Delta tk)}{k^2}\,dk \\ &= \begin{cases} \frac{1}{R^2}\left[\frac{R}{4}\cos(\arcsin(\Delta t/R)) + \frac{R}{12}\cos(3\arcsin(\Delta t/R))\right] & \text{for } R^2 > \Delta t^2\,, \\ 0 & \text{for } R^2 < \Delta t^2\,. \end{cases} \\ &= \begin{cases} \left(\frac{1}{3} - \frac{\Delta t^2}{3R^2}\right)\frac{\sqrt{R^2-\Delta t^2}}{R^2} = \left(\frac{1}{3} - \frac{\Delta t^2}{3R^2}\right)S(0) & \text{for } R^2 > \Delta t^2\,, \\ 0 & \text{for } R^2 < \Delta t^2\,. \end{cases} \end{aligned} \tag{A20}$$

In the above calculation, we have made use of the following trigonometric relations

$$\cos(3x) = 4\cos^3(x) - 3\cos(x), \qquad \cos(\arcsin x) = \sqrt{1-x^2}. \tag{A21}$$

Therefore one finds, using the recursive relation Eq (A18),

$$\begin{aligned} Q(m) &= \frac{(2m)!!}{2R^{2m-2}}Q(1) - \frac{1}{R^2}\sum_{k=2}^{m}\frac{(2m)!!}{(2k)!!}\frac{S(k-2)}{R^{2m-2k}} \\ &= \frac{(2m)!!}{R^{2m-2}}\left[\frac{1}{2}Q(1)\right. \\ &\quad \left. + \sum_{k=2}^{m}\sum_{j=0}^{k-2}\frac{(2j+1)!!}{2k(2k-2)(2j)!!(2j+1)(2j-1)}\frac{R^{2j}}{(R^2-\Delta t^2)^j}S(0)\right]. \end{aligned} \tag{A22}$$

This expression can be simplified if we note that

$$\sum_{k=j+2}^{m}\frac{1}{k(k-1)}=\sum_{k=2}^{m}\frac{1}{k(k-1)}-\sum_{k=2}^{j+1}\frac{1}{k(k-1)}=\frac{m-j-1}{m(j+1)}\,, \tag{A23}$$

and

$$\begin{aligned}&\sum_{k=2}^{m}\sum_{j=0}^{k-2}\frac{(2j+1)!!}{2k(2k-2)(2j)!!(2j+1)(2j-1)}\frac{R^{2j}}{(R^2-\Delta t^2)^j}S(0)\Big]\\&=\sum_{j=0}^{m-2}\sum_{k=j+2}^{m}\frac{(2j+1)!!}{2k(2k-2)(2j)!!(2j+1)(2j-1)}\frac{R^{2j}}{(R^2-\Delta t^2)^j}S(0)\Big]\\&=\sum_{j=0}^{m-2}\frac{(m-j-1)(2j+1)!!}{4m(j+1)(2j)!!(2j+1)(2j-1)}\frac{R^{2j}}{(R^2-\Delta t^2)^j}S(0)\Big]\end{aligned} \tag{A24}$$

So, we have in this case

$$D^{2m+1}=\begin{cases}\frac{(2m+1)!!}{2(2\pi)^{m+2}}\frac{1}{(R^2-\Delta t^2)^{m+\frac{3}{2}}}\,, & \text{for } R^2>\Delta t^2\,,\\ 0 & \text{for } R^2<\Delta t^2\,,\end{cases} \tag{A25}$$

$$F^{2m+1}_{ij}=-\frac{1}{2(2\pi)^{m+2}}\,\partial_i\partial'_j\left(\frac{(2m)!!}{R^{2m}}S(0)\sum_{k=0}^{m}\frac{(2k+1)!!}{(2k)!!(2k+1)(2k-1)}\frac{R^{2k}}{(R^2-\Delta t^2)^k}\right)\,, \tag{A26}$$

and

$$\begin{aligned}H^{2m+1}_{ijkl}=&\frac{1}{2(2\pi)^{m+2}}\,\partial_i\partial'_j\partial_k\partial'_l\Big\{\frac{(2m)!!}{R^{2m-2}}\Big[\frac{1}{2}Q(1)\\&+\sum_{j=0}^{m-2}\frac{(m-j-1)(2j+1)!!}{4m(j+1)(2j)!!(2j+1)(2j-1)}\frac{R^{2j}}{(R^2-\Delta t^2)^j}S(0)\Big]\,.\end{aligned} \tag{A27}$$

B. The case of even n

Let $n=2m$ with $m=1,2,3....$ The graviton two-point functions for $m=0$ corresponding to the usual 4 dimensional spacetime have been given previously [11]. The analog of Eq. (A11) for this case is

$$F^{2m}_{ij}=\frac{1}{2(2\pi)^{m+\frac{3}{2}}}\,\partial_i\partial'_j\Big(S(m-\frac{1}{2})\Big)\,. \tag{A28}$$

Here

$$S(m-1/2)=\frac{2m-1}{R^2}S(m-\frac{3}{2})-T(m-\frac{3}{2})\,. \tag{A29}$$

Using this recursive relation, we can express $S(m-1/2)$ in terms of S(1/2) which is calculated, by employing

$$J_{n+\frac{1}{2}}(z) = (-1)^n\, z^{n+\frac{1}{2}} \sqrt{\frac{2}{\pi}} \frac{d^n}{(zdz)^n}\Big(\frac{\sin z}{z}\Big)\,, \tag{A30}$$

to be

$$\begin{aligned} S(1/2) &= \frac{1}{R^{\frac{3}{2}}} \int_0^\infty k^{-1/2} J_{\frac{3}{2}}(Rk) \cos(\Delta t k)\, dk \\ &= -\sqrt{\frac{2}{\pi}} \frac{1}{R^3} \int_0^\infty \frac{d}{dk}\Big(\frac{\sin(Rk)}{k}\Big) \cos(\Delta t k)\, dk \\ &= -\sqrt{\frac{2}{\pi}} \left(\frac{1}{R^3} \frac{\sin(Rk)\cos(\Delta t k)}{k}\bigg|_0^\infty + \frac{\Delta t}{R^3} \int_0^\infty \frac{\sin(Rk)\sin(\Delta t k)}{k}\, dk \right) \\ &= \sqrt{\frac{2}{\pi}} \left(\frac{1}{R^2} - \frac{\Delta t}{4R^3} \ln\Big(\frac{R+\Delta t}{R-\Delta t}\Big)^2 \right)\,. \end{aligned} \tag{A31}$$

It then follows that

$$\begin{aligned} S(m-1/2) &= \frac{(2m-1)!!}{R^{2m-2}} S(1/2) - \sum_{k=2}^{m} \frac{(2m-1)!!}{(2k-1)!!} \frac{T(k-3/2)}{R^{2m-2k}} \\ &= \frac{(2m-1)!!}{R^{2m}} \sqrt{\frac{2}{\pi}} \left[1 - \frac{\Delta t}{4R} \ln\Big(\frac{R-\Delta t}{R-\Delta t}\Big)^2 - \frac{1}{R^2} \sum_{k=2}^{m} \frac{2^{k-2}\Gamma(k-1)}{(2k-1)!!} \frac{R^{2k}}{(R^2-\Delta t^2)^{k-1}} \right]\,. \end{aligned} \tag{A32}$$

Similarly, one has for H^{2m}_{ijkl}

$$H^{2m}_{ijkl} = \frac{1}{2(2\pi)^{m+\frac{3}{2}}} \partial_i \partial'_j \partial_k \partial'_l \Big(Q(m-\frac{1}{2}) \Big)\,, \tag{A33}$$

and

$$Q(m-1/2) = \frac{2m-1}{R^2} Q(m-\frac{3}{2}) - \frac{1}{R^2} S(m-\frac{5}{2})\,. \tag{A34}$$

Now the calculation becomes a little tricky. First, let us note that H^0_{ijkl} has already been given [11] and the recursive relation Eq (A34) can only be applied when $m \geq 3$. So, we need both H^2_{ijkl} and H^4_{ijkl} or $Q(1/2)$ and $Q(3/2)$ as our basis to use the recursive relation for a general expression. Because there is an infrared divergence in the $Q(1/2)$ integral, so, as we did in the 4 dimensional case, we will introduce a regulator β in the denominator of the integrand and then let β approach 0 after the integration is performed. Noting that

$$J_{\frac{3}{2}}(z) = \sqrt{\frac{2}{\pi z}} \Big(\frac{\sin z}{z} - \cos z \Big)\,, \tag{A35}$$

we obtain

$$
\begin{aligned}
Q(1/2) &= \frac{1}{R^{\frac{3}{2}}} \int_0^\infty k^{\frac{-5}{2}} J_{\frac{3}{2}}(kR)\cos(kt)\,dk \\
&= \sqrt{\frac{2}{\pi}}\Big(\frac{1}{R^3}\int_0^\infty \frac{dk}{k^4}\sin kR\,\cos k\Delta t \;-\frac{1}{R^2}\int_0^\infty \frac{dk}{k^3}\cos kR\,\cos k\Delta t\Big) \\
&= \sqrt{\frac{2}{\pi}}\lim_{\beta\to 0}\Big(-\frac{1}{R^3}\frac{1}{2\beta}\frac{\partial}{\partial\beta}\int_0^\infty \frac{\sin kR\cos k\Delta t}{k^2+\beta^2}\,dk \\
&\quad +\frac{1}{R^2}\frac{1}{2\beta}\frac{\partial}{\partial\beta}\int_0^\infty \frac{k\cos kR\cos k\Delta t}{k^2+\beta^2}\,dk\Big)\,.
\end{aligned}
\tag{A36}
$$

We next use

$$
\begin{aligned}
\int_0^\infty \frac{\sin(ax)\cos(bx)}{\beta^2+x^2}\,dx &= \frac{1}{4\beta}e^{-a\beta}\{e^{b\beta}Ei[\beta(a-b)]+e^{-b\beta}Ei[\beta(a+b)]\} \\
&\quad -\frac{1}{4\beta}e^{a\beta}\{e^{b\beta}Ei[-\beta(a+b)]+e^{-b\beta}Ei[-\beta(a-b)]\}\,,
\end{aligned}
\tag{A37}
$$

$$
\begin{aligned}
\int_0^\infty \frac{x\cos(ax)\cos(bx)}{\beta^2+x^2}\,dx &= -\frac{1}{4}e^{-a\beta}\{e^{b\beta}Ei[\beta(a-b)]+e^{-b\beta}Ei[\beta(a+b)]\} \\
&\quad -\frac{1}{4}e^{a\beta}\{e^{b\beta}Ei[-\beta(a+b)]+e^{-b\beta}Ei[-\beta(a-b)]\}\,,
\end{aligned}
\tag{A38}
$$

where Ei(x) is the exponential-integral function, and the fact that, when x is small,

$$
\mathrm{Ei(x)} \approx \gamma + \ln|x| + x + \frac{1}{4}x^2 + \frac{1}{18}x^3 + O(x^4)\,,
\tag{A39}
$$

where γ is the Euler constant. After expanding $Q(1/2)$ around $\beta = 0$ to the order of β^2, one finds

$$
\begin{aligned}
Q(1/2) = \lim_{\beta\to 0}\sqrt{\frac{2}{\pi}}\Big(&\frac{5}{18} - \frac{1}{3}\gamma - \frac{1}{3}\ln(\beta) - \frac{1}{6}\ln(R^2-\Delta t^2) \\
&-\frac{\Delta t^2}{6R^2} + \frac{\Delta t}{8R}\Big(\frac{\Delta t^2}{3R^2}-1\Big)\ln\Big(\frac{R+\Delta t}{R-\Delta t}\Big)^2\Big)\,.
\end{aligned}
\tag{A40}
$$

Note, however, that what we need is H_{ijkl} which involves differentiation of $Q(1/2)$, therefore we can discard the constant and divergent terms in $Q(1/2)$ as far as H_{ijkl} is concerned. To calculate $Q(3/2)$, let us recall that

$$
Q(3/2) = \frac{3}{R^2}Q(1/2) - \frac{1}{R^2}S(-1/2)
\tag{A41}
$$

and note that $S(-1/2)$ is given by $\sqrt{2/\pi}$ times Eq (A19) in Ref. [11] Thus, we have

$$
Q(3/2) = \sqrt{\frac{2}{\pi}}\left(-\frac{1}{6R^2} - \frac{\Delta t^2}{2R^4} + \frac{\Delta t}{8R^3}\Big(\frac{\Delta t^2}{R^2}-1\Big)\ln\Big(\frac{R+\Delta t}{R-\Delta t}\Big)^2\right)\,.
\tag{A42}
$$

With $Q(3/2)$ at hand, it is easy to show that for an arbitrary $m \geq 3$

$$
\begin{aligned}
Q(m-1/2) &= \frac{(2m-1)!!}{3R^{2m-4}}Q(3/2) - \frac{1}{R^2}\sum_{k=3}^{m}\frac{(2m-1)!!}{(2k-1)!!}\frac{S(k-5/2)}{R^{2m-2k}} \\
&= \frac{(2m-1)!!}{R^{2m-4}}\bigg(\frac{1}{3}Q(3/2) - \sum_{k=3}^{m}\frac{1}{(2k-1)(2k-3)}S(1/2) \\
&\quad + \frac{1}{R^4}\sqrt{\frac{2}{\pi}}\sum_{k=3}^{m}\sum_{j=2}^{k-2}\frac{2^{j-2}\Gamma(j-1)}{(2j-1)!!(2k-1)(2k-3)}\frac{R^{2j}}{(R^2-\Delta t^2)^{j-1}}\bigg) \\
&= \frac{(2m-1)!!}{R^{2m-4}}\bigg(\frac{1}{3}Q(3/2) - \sum_{k=3}^{m}\frac{1}{(2k-1)(2k-3)}S(1/2) \\
&\quad + \frac{1}{R^4}\sqrt{\frac{2}{\pi}}\sum_{j=2}^{m-2}\frac{(m-j-1)2^{j-2}\Gamma(j-1)}{(2m-1)(2j+1)!!}\frac{R^{2j}}{(R^2-\Delta t^2)^{j-1}}\bigg).
\end{aligned}
\tag{A43}
$$

Here in the last step, we have made use of the following results

$$
\begin{aligned}
\sum_{k=j+2}^{m}\frac{1}{(2k-1)(2k-3)} &= \sum_{k=2}^{m}\frac{1}{(2k-1)(2k-3)} - \sum_{k=2}^{j+1}\frac{1}{(2k-1)(2k-3)} \\
&= \frac{m-j-1}{(2m-1)(2j+1)},
\end{aligned}
\tag{A44}
$$

and

$$
\sum_{k=3}^{m}\sum_{j=2}^{k-2} f(j)g(k) = \sum_{k=4}^{m}\sum_{j=2}^{k-2} f(j)g(k) = \sum_{j=2}^{m-2} f(j)\sum_{k=2+j}^{m} g(k)\,. \tag{A45}
$$

Consequently, we obtain

$$
D^{2m} = \frac{2^m m!}{(2\pi)^{m+2}}\frac{1}{(R^2-\Delta t^2)^{m+1}}, \tag{A46}
$$

$$
\begin{aligned}
F_{ij}^{2m} &= \frac{1}{(2\pi)^{m+2}}\,\partial_i\partial'_j\bigg\{\frac{(2m-1)!!}{R^{2m}}\bigg[1 - \frac{\Delta t}{4R}\ln\Big(\frac{R-\Delta t}{R-\Delta t}\Big)^2 \\
&\quad - \frac{1}{R^2}\sum_{k=2}^{m}\frac{2^{k-2}\Gamma(k-1)}{(2k-1)!!}\frac{R^{2k}}{(R^2-\Delta t^2)^{k-1}}\bigg]\bigg\},
\end{aligned}
\tag{A47}
$$

and

$$
\begin{aligned}
H_{ijkl}^{2m} = \frac{1}{(2\pi)^{m+2}}\,\partial_i\partial'_j\partial_k\partial'_l\bigg\{&\frac{(2m-1)!!}{R^{2m-4}}\bigg[\frac{\Delta t}{24R^3}\Big(\frac{\Delta t^2}{R^2}-1\Big)\ln\Big(\frac{R+\Delta t}{R-\Delta t}\Big)^2 - \frac{1}{18R^2} \\
&-\frac{\Delta t^2}{6R^4} - \sum_{k=3}^{m}\frac{1}{(2k-1)(2k-3)}\bigg(\frac{1}{R^2} - \frac{\Delta t}{4R^3}\ln\Big(\frac{R+\Delta t}{R-\Delta t}\Big)^2\bigg) \\
&+\frac{1}{R^4}\sum_{j=2}^{m-2}\frac{(m-j-1)2^{j-2}\Gamma(j-1)}{(2m-1)(2j+1)!!}\frac{R^{2j}}{(R^2-\Delta t^2)^{j-1}}\bigg).
\end{aligned}
\tag{A48}
$$

REFERENCES

[1] Th. Kaluza, Sitzungsber. Preuss. Akad. Wiss., Phys. Math. Kl., 996 (1921); O. Klein, Z. Phys. **37**, 895 (1926).

[2] I. Antoniadis, Phys. Lett. **B246**, 377 (1994).

[3] V.A. Kostelecky and S. Samuel, Phys. Lett. **B270**, 21 (1991).

[4] I. Antoniadis and K. Benakli, Phys. Lett. **B326**, 69 (1994).

[5] N. Arkani-Hamed, S. Dimopoulos and G. Dvali, Phys. Lett. **B429**, 263 (1998); I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos and G. Dvali, Phys. Lett. **B436**, 247 (1998); N. Arkani-Hamed, S. Dimopoulos and G. Dvali, Phys. Rev. **D59**, 086004 (1999).

[6] E. Witten, Nucl. Phys. **B471**, 135 (1996); J, Lykken, Phys. Rev. **D 54**, 3693 (1996); G. Shiu and S.-H.H. Tye, Phys. Rev. **D 58**, 106007 (1998).

[7] J.L. Hewett, Phys. Rev. Lett. **82**, 4765 (1999);
G.F. Giudice, R. Rattazzi and J.D. Wells, Nucl. Phys. **B544**, 3 (1999);
T.G. Rizzo, Phys. Rev. **D59**, 115010 (1999);
I. Antoniadis, K. Benakli, and M. Quiros, *Direct collider signatures of large extra dimensions*, hep-ph/9905311;
T. G. Rizzo and J. D. Wells, Phys. Rev. **D61**, 016007 (2000) .

[8] A. Kehagias and K. Sfetsos, *Deviations from the* $1/r^2$ *Newton law due to extra dimensions*, hep-ph/9905417;
E.G. Floratos and G.K. Leontaris, Phys. Lett. **B465**, 95 (1999), hep-ph/9906238.

[9] K.R. Dienes, E.Dudas, and T. Ghergheta, Phys. Lett. **436**, 55 (1998), hep-ph/9803466; Nucl. Phys. **537**, 47(1999), hep-ph/9806292;
D. Chilencea and G.G. Ross, Phys. Lett. **B442**, 165(1998), hep-ph/9809217;
S.A. Abel and S.F. King, Phys. Rev. **D59**, 095010 (1999), hep-ph/9809467.
C.D. Carrone, Phys. Lett. **B454**, 70 (1999), hep-ph/9902407.
A. Delgado and Quiros, Nucl. Phys. **B559**, 235 (1999), hep-ph/9903400.
A. Perez-Lorenzana and R. N. Mohapatra, Nucl. Phys. **B559**, 255 (1999), hep-ph/9904504.
H-C. Cheng, B. A. Dobrescu and C. T. Hill, *Gauge Coupling Unification with Extra Dimensions and Gravitational Scale Effects*, hep-ph/9906327.

[10] N. Kaloper and A. Linde, Phys. Rev. D**59**, 101303 (1999);
D. Lyth, Phys. Lett., B**448**, 191(1999);
G. Dvali, and S.-H. Tye, Phys. Lett., B**450**, 72(1999);
C. Csaki, M. Graesser and J. Terning, Phys. Lett., B**456**, 16(1999);
G. Dvali, Phys. Lett., B**459**, 489(1999);
N. Arkani-Hamed, S. Dimopoulos, N. Kaloper and John March-Russell, *Rapid Asymmetric Inflation and Early Cosmology in Theories with Sub-Millimeter Dimensions*, hep-ph/9903224;
A. Mazumdar, Phys. Lett. **B469**, 55 (1999), hep-ph/9902381;
L. J. Hall and D. Smith, Phys. Rev. **D60**, 085008 (1999), hep-ph/9904267;
A. Riotto, *D-branes, String Cosmology and Large Extra Dimensions*, hep-ph/9904485;
J. M. Cline, Phys. Rev. **D61**, 023513 (2000), hep-ph/9904495;
A. Melchiorri, F. Vernizzi, R. Durrer, and G. Veneziano, Phys. Rev. Lett. **83**, 4464 (1999), astro-ph/9905327.

[11] H. Yu and L.H. Ford, Phys. Rev. D**60**, 084023 (1999), gr-qc/9904082.

[12] H. Yu and L.H. Ford, *Lightcone fluctuations in quantum gravity and extra dimensions*, gr-qc/9907037.

[13] T. Appelquist and A. Chodos, Phys. Rev. Lett. **50**, 141 (1983); Phys. Rev. **D28**, 772 (1983).

[14] M.A. Rubin and B. Roth, Phys. Lett. **127B**, 55 (1983); K. Tsokos, Phys. Lett. **126B**, 451 (1983).

[15] L.H. Ford, Phys. Rev. **D51**, 1692 (1995), gr-qc/9410043.

[16] C.-H. Wu and L.H. Ford, Phys. Rev. D **60**, 104013 (1999), gr-qc/9905012.

[17] L.H. Ford and N. F. Svaiter, Phys. Rev. **D54**, 2640 (1996), gr-qc/9604052

[18] G. Amelino-Camelia, J. Ellis, N.E. Mavromatos and D.V. Nanopoulos, Int. J. Mod. Phys. **A12**, 607 (1997).

[19] J. Ellis, N.E. Mavromatos and D.V. Nanopoulos, *Quantum-Gravitational Diffusion and Stochastic Fluctuations in the Velocity of Light*, gr-qc/9904068; gr-qc/9905048.

[20] E.W. Kolb, M.J. Perry, and T.P. Walker, Phys. Rev. **D33**, 869 (1986).

[21] J.D. Barrow, Phys. Rev. **D35**, 1805 (1987).

[22] M. Maurette, Ann. Rev. Nucl. Sci. **26**, 319 (1976).

[23] A.I. Shylakhter, Nature (London), **264**, 340 (1976).
Phys. **A12**, 607 (1997).

[24] D.B. Sanders et al, Astr. Ap., **215**, L5 (1989).

[25] J.C. Long, H.W. Chan, and J.C. Price, Nucl. Phys. **B539**, 23 (1999), hep-ph/9805217.

[26] T. Piran. Phys. Rep. **314**, 575 (1999).

[27] M. Rees , astro-ph/9701162.

[28] G. Amelino-Camelia, J. Ellis, N.E. Mavromatos, D.V. Nanopoulos and S. Sarkar, Nature **393**, 763 (1998).

[29] B.E. Schaefer, Phys. Rev. Lett., **82**, 4964 (1999).

[30] S.D. Biller et al, Phys. Rev. Lett., **83**, 2108 (1999).

[31] P. Kaaret, Astronomy and Astrophysics, **345**, L3 (1999).

[32] J. Ellis et al, *Astrophysical probes of the constancy of the velocity of light* , astro-ph/9907340.

[33] P.N. Bhat *et al.,* Nature(London) **359**, 217 (1992).

[34] K. Hurley *et al.,* Nature(London) **372**, 652 (1994).

[35] M. Sommer *et al.,* Astrophys. J. **422**, L63 (1994).

[36] S.R. Kulkarni, *et al.* Nature **395** 663 (1998).

[37] S.R. Kulkarni, *et al.* Nature **398** 389 (1999); T.J. Galama, *et al.* Nature **398** 394 (1999); C. Akerhof, *et al.* Nature **398** 400 (1999).

[38] B.S. DeWitt, in *Relativity, Groups and Topology*, eds. B.S. and C.M. de Witt (Gordon and Breach, New York, 1963).

[39] N.E. Mavromatos and R. Szabo, Phys. Rev. **D59**, 104018 (1999).

[40] A. Campbell-Smith, J. Ellis, N.E. Mavromatos and D.V. Nanopoulos, Phys. Lett. **B466**, 11 (1999), hep-th/9907141.

[41] A. Abramovici et al., Science 256, 325(1992).

[42] C. Bradaschia et al., Nucl. Instrum. Meth. **289A**, 518 (1990).

[43] A. Abramovici et al., Phys. Lett., **218A**, 157 (1996).

[44] G. Amelino-Camelia, Nature **398**, 216 (1999); *On the Salecker-Wigner limit and the use of interferometers in space-time-foam studies*, gr-qc/9910023; *Are we at the dawn*

of quantum-gravity phenomenology?, gr-qc/9910089.
[45] Y.J. Ng and H. van Dam, Phys.Lett. **B477**, 429 (2000); gr-qc/9906003.
[46] L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, 4690 (1999).
[47] I.S. Gradshteyn and I.M. Ryzhik, *Table of Integrals, Series, and Products*, (Academic Press, New York, 1965).